Recent Results on Integrable and Non-Integrable Lotka Volterra Systems

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Contents of the Lecture (*)

1. Integrability of the Antisymmetric Lotka Volterra Hamiltonian system (LVH) of competing species without linear terms.
2. Integrability of the Antisymmetric LVH system with linear terms.
3. LVH systems exhibit very simple dynamics even when we strongly perturb them away from integrability.
4. Interesting connections between Painlevé conditions for integrability and the approach of Quasi-Polynomial Canonical Forms.

(*) Based on the paper, “Comparison Between the QP Formalism and the Painlevé Property in Integrable Dynamical Systems”, subm. to TMP, 2022)

What does Integrability mean?

It means our ability to integrate systems of Ordinary Differential Equations (ODEs), Partial Differential Equations (continuous variables), or Difference Equations (Discrete variables, ΔEs or PΔEs).

Does this mean we can then solve these equations?

Not necessarily! This may still be quite a difficult task!

There are many sophisticated methods to establish integrability:

a) If the system of ODEs is Hamiltonian, we can look for Action – Angle variables, Symmetries, or Lax Pairs
b) If we study PDEs, we can use Inverse Scattering Transforms, Lax Pairs, Bäcklund transformations
c) Many Integrable PΔEs have also been discovered using methods inspired from integrable PDEs.

What is the Painlevé property for ODEs?

It is a method that requires the general (unknown) solutions of a given system of ODES to be single-valued in the complex plane of their independent variable (i.e. time), i.e. they can only have pole singularities of the form:

$$(t-t_*)^{-p}, \quad p = positive\, integer$$

t_* being the (arbitrary) location of the singularity. This means that all solutions are Laurent series expansions of the form:

$$x_i(t) = \frac{1}{\tau^p}\sum_{k=0}^{\infty} a_{i,k}\tau^k, \quad i = 1, \ldots, n, \qquad \tau = t - t_*$$

with n-1 arbitrary constants among the $a_{i,k}$ (t_* is the nth).

If this can be done for some values of the parameters, then this is a necessary condition that for these parameter values the original system of ODEs is integrable!

Consider or example the famous Hénon Heiles system, which was studied by this method in Bountis et al., Phys. Rev. A25 , 1257 (1982). Its equations of motion:

$$\frac{d^2x_1}{dt^2} = -Ax_1 - 2x_1x_2, \qquad \frac{d^2x_2}{dt^2} = -Bx_2 - x_1^2 + Cx_2^2$$

describe the (Newtonian) orbits of a particle in an axisymmetric 3 - dimensional potential.

Using the Painlevé property it was discovered that the HH system may be integrable in only 3 cases:

Case 1 : A = B, C = −1 (separable in x_1-x_2 , x_1+x_2 variables)
Case 2 : A,B free, C = −6
Case 3 : B = 16A, C = −16

The second integral (the first is the Hamiltonian) has also been found in Cases 2 and 3, while no other integrable cases are known to date.

1. INTEGRABILITY OF LOTKA VOLTERRA SYSTEMS

Let us consider a class of Lotka Volterra Hamiltonian systems which preserve the Hamiltonian

$$H(x_1,....,x_n)=\sum_{i=1}^{n}x_i=h=const., \qquad (1)$$

in n dimensions, whose equations of motion possess the Poisson structure

$$\frac{dx_i}{dt}=\dot{x}_i=\{x_i,x_j\}=\sum_{i=1}^{n}a_{ij}x_ix_j \qquad i=1,...,n \qquad (2)$$

where $A=(a_{ij})$ is an antisymmetric matrix with

$$a_{ij}=-a_{ji} \qquad i,j=1,...,n$$

Let us investigate the integrability properties of this system using the Painlevé (P-) property. We shall start with the 3-dimensional case:

1. THE ANTISYMMETRIC LOTKA VOLTERRA SYSTEM

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Let us investigate the integrability of this system using the Painlevé (P-) property. We shall start with the 3-dimensional case:

1a) The case n=3:

The equations of motion are given by:

$$\dot{x}_1 = x_1(Ax_2 + Bx_3)$$
$$\dot{x}_2 = x_2(-Ax_1 + Dx_3) \qquad (3)$$
$$\dot{x}_3 = x_3(-Bx_1 - Dx_2)$$

We expand our solutions about any "movable" (i.e. initial condition dependent) singularity of the solutions $t = t_*$ in Laurent series, in the variable $\tau = t - t_*$:

$$x_1 = \alpha\tau^{-1} + a_0 + a_1\tau + ... + a_{r-1}\tau^{r-1}...,$$
$$x_2 = \beta\tau^{-1} + b_0 + b_1\tau + ... + b_{r-1}\tau^{r-1}..., \qquad (4)$$
$$x_3 = \gamma\tau^{-1} + c_0 + c_1\tau + ... + c_{r-1}\tau^{r-1}...,$$

Where t_* is the first free constant and we seek 2 more free constants to obtain the expansions of the general solution.

Substituting (4) in (3) we find that the leading order coefficients satisfy:

$$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & A & B \\ -A & 0 & D \\ -B & -D & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = M_3 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \qquad (5)$$

The determinant of the antisymmetric matrix M_3 is zero, hence a second free constant enters the series (10) at leading order iff the lhs of the above equations satisfy the necessary orthogonality conditions. These imply that a solution of (5) exists, iff

$$B = A + D \qquad (6)$$

Using (4) and (3), we now equate coefficients of the general terms a_{r-1} , b_{r-1} c_{r-1} , and obtain a homogeneous system, where arbitrary constants enter at order r satisfying the resonance condition:

$$r^3 - r = (r+1)r(r-1) = 0 \qquad (7)$$

Where r=-1 corresponds to the freedom of t_* , r=0 to a second free constant at leading order and r = 1 to the 3rd free constant at first order.

We then find at order r = 1, that a 3d free constant enters among the a_0 , b_0 ,c_0 coefficients in

$$\begin{pmatrix} -1 & A\alpha & B\alpha \\ -A\beta & -1 & D\beta \\ -B\gamma & -D\gamma & -1 \end{pmatrix}\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = N_3\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad , \qquad (8)$$

since $\det N_3 = 0$, provided $B = A+D$.

We now verify that the above conditions imply the complete integrability and solvability of the equations of motion:

Using first eq. (1) to write $x_3 = h - x_1 - x_2$ we obtain from the first 2 equations in (3):

$$\begin{aligned} \dot{x}_1 &= x_1[Bh+(A-B)x_2 - Bx_1] \\ \dot{x}_2 &= x_2[(B-A)h - Bx_1 + (A-B)x_2] \end{aligned} \qquad (9)$$

Dividing these equations by x_1 and x_2 respectively and subtracting them by sides we obtain:

$$\frac{\dot{x}_1}{x_1} - \frac{\dot{x}_2}{x_2} = Bh - (B-A)h = Ah \qquad (10)$$

which is readily integrated to

$$(x_1 / x_2)\,exp(-Aht) = K = arbitr.\,const.$$

…thereby arriving at the complete solution of the problem.

1b) The case n=4

We now repeat this approach starting with the equations

$$\begin{aligned} \dot{x}_1 &= x_1(Ax_2 + Bx_3 + Cx_4) \\ \dot{x}_2 &= x_2(-Ax_1 + Dx_3 + Ex_4) \\ \dot{x}_3 &= x_3(-Bx_1 - Dx_2 + Fx_4) \\ \dot{x}_4 &= x_4(-Cx_1 - Ex_2 - Fx_3) \end{aligned} \qquad (11)$$

where the condition for finding free constants becomes

$$r^4 - r^2 = r^2(r+1)(r-1) = 0 \qquad (12)$$

and has the same structure as before. So, we finally find here 3 restraining relations on the coefficients of (11):

$$B = A + D, \qquad F = C - B, \qquad C = A + E$$

We demonstrate in this way that there are 2 free constants at leading order as predicted by (12) and hence the P-property is established provided the above 3 relations hold.

We thus say that our system satisfies what we call the strong P-property if the following holds:

(a) The system has solutions where each variable is expressed as a Laurent series that starts with a simple pole, and

(b) n-2 free parameters appear at the leading order term of these expansions, by proving that general resonance condition for all n is:

$$r^n - r^{n-2} = r^{n-2}(r+1)(r-1) = 0 \qquad (13)$$

Of the n(n-1)/2 parameters entering the original equations of motion (2), n-1 are free and hence there are (n-1)(n-2)/2 restraining relations.

1c) The case of arbitrary n

We have proved (Bountis and Vanhaecke, 2016) that the general n – dimensional LVH system (1) and (2) with the antisymmetric matrix $A = (a_{i,j})$: $a_{ij} = -a_{ji} \qquad i, j = 1,\ldots,n$

$$\begin{aligned}
\dot{x}_1 \;&= x_1(a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n-1}x_{n-1} + a_{1,n}x_n) \\
\dot{x}_2 \;&= x_2(-a_{1,2}x_1 + a_{2,3}x_3 + \ldots + a_{2,n-1}x_4 + a_{2,n}x_n) \\
&\ldots. \\
\dot{x}_{n-1} &= x_{n-1}(-a_{1,n-1}x_1 - a_{2,n-1}x_2 - \ldots - a_{n-2,n-1}x_{n-2} + a_{n,n-1}x_n) \\
\dot{x}_n \;&= x_n(-a_{1,n}x_1 - a_{2,n}x_2 - \ldots. - a_{n-2,n}x_3 - a_{n-1,n}x_{n-1})
\end{aligned} \qquad (14)$$

has the strong Painlevé property, and is completely integrable, if the coefficients in (14) satisfy the relations:

$$a_{i,j} = a_j - a_i, \quad j = 2,3,\ldots,n, \quad i = 1,2,\ldots, j-1 \qquad (15)$$

for n – 1 free constants a_j . We have also shown that these equations can be written in the Lax form, dL/dt =[L,M], for two appropriate matrices L and M.

To prove that we can completely integrate eq. (14) we express the last variable x_n in terms of all the others, using the integral (1), whence system (14) becomes:

$$
\begin{aligned}
\dot{x}_1 &= x_1(a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n-1}x_{n-1} + a_{1,n}(h - x_1 - x_2 - \ldots - x_{n-1})) \\
\dot{x}_2 &= x_2(-a_{1,2}x_1 + a_{2,3}x_3 + \ldots + a_{2,n-1}x_4 + a_{2,n}(h - x_1 - x_2 - \ldots - x_{n-1})) \\
&\ldots. \\
\dot{x}_{n-1} &= x_{n-1}(-a_{1,n-1}x_1 - a_{2,n-1}x_2 - \ldots - a_{n-2,n-1}x_{n-2} + a_{n-1,n}(h - x_1 - x_2 - \ldots - x_{n-1})) \\
\dot{x}_n &= x_n(-a_{1,n}x_1 - a_{2,n}x_2 - \ldots. - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1})
\end{aligned}
\qquad (16)
$$

Rearranging terms in (16) and using (15) we rewrite these equations as:

$$
\begin{aligned}
\dot{x}_1 &= x_1(a_{1,n}h - a_{2,n}x_2 - a_{3,n}x_3 + \ldots - a_{n-1,n}x_{n-1} - a_{1,n}x_1) \\
\dot{x}_2 &= x_2(a_{2,n}h - a_{1,n}x_1 - a_{3,n}x_3 + \ldots - a_{n-1,n}x_{n-1} - a_{2,n}x_2) \\
&\ldots. \\
\dot{x}_{n-1} &= x_{n-1}(a_{n-1,n}h - a_{1,n}x_1 - a_{2,n}x_2 - \ldots - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1}) \\
\dot{x}_n &= x_n(-a_{1,n}x_1 - a_{2,n}x_2 - \ldots. - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1})
\end{aligned}
\qquad (17)
$$

We now observe that the quantity

$$Q = \frac{\dot{x}_n}{x_n} = -a_{1,n}x_1 - a_{2,n}x_2 - .... - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1} \qquad (18)$$

is present in every equation in (17) and hence these equations become:

$$\frac{\dot{x}_k}{x_k} = a_{k,n}h + Q = a_{k,n}h + \frac{\dot{x}_n}{x_n}, \qquad \mathrm{k} = 1, 2, ..., \mathrm{n} - 1 \qquad (19)$$

Eqns. (29) are integrated to yield $x_k(t)$ as function of $x_n(t)$:

$$x_k(\mathrm{t}) = c_k x_n(\mathrm{t})\exp(a_{k,n}ht), \quad k = 1,2,....,n - 1 \qquad (20)$$

with n – 1 arbitrary constants c_k and n – 1 free parameters. The nth variable is then evaluated using the above expressions:

$$\frac{\dot{x}_n}{x_n} = -\sum_{k=1}^{n-1} a_{k,n} x_k(t) = -x_n(t) \sum_{k=1}^{n-1} c_k a_{k,n} \, exp(a_{k,n} ht) \quad \Rightarrow \frac{\dot{x}_n}{x_n^2} = -\sum_{k=1}^{n-1} c_k a_{k,n} \, exp(a_{k,n} ht)$$

$$\Rightarrow \frac{1}{x_n} = \sum_{k=1}^{n-1} c_k a_{k,n} \int_0^t exp(a_{k,n} ht) dt + c_n = \frac{1}{h} \sum_{k=1}^{n-1} c_k \left(exp(a_{k,n} ht) - 1 \right) + c_n$$

Whence,

$$x_n(t) = \frac{h}{\sum_{k=1}^{n-1} c_k \left(exp(a_{k,n} ht) - 1 \right) + c_n h} \qquad \textbf{(21)}$$

and c_n plays the role of an nth arbitrary constant. Using the Hamiltonian integral, and (21) we find that the final form for $x_n(t)$ is

$$x_n(t) = \frac{h}{\sum_{k=1}^{n-1} c_k \, exp(a_{k,n} ht) + 1} \qquad \textbf{(22)}$$

Comparing (21) and (22) we conclude that, $-\sum_{k=1}^{n-1} c_k + c_n h = 1$ and the arbitrary constants are again n – 1.

2. Integrable LV Antisymmetric Systems with Linear Terms

An interesting question arises now about an LVH system including linear terms:

$$\begin{aligned}
\dot{x}_1 &= b_{11}x_1 + \ldots + b_{1n}x_n + x_1(a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n-1}x_{n-1} + a_{1,n}x_n) \\
\dot{x}_2 &= b_{21}x_1 + \ldots + b_{2n}x_n + x_2(-a_{1,2}x_1 + a_{2,3}x_3 + \ldots + a_{2,n-1}x_4 + a_{2,n}x_n) \\
&\ldots. \\
\dot{x}_{n-1} &= b_{n-1,1}x_1 + \ldots + b_{n-1,n}x_n + x_{n-1}(-a_{1,n-1}x_1 - a_{2,n-1}x_2 - \ldots - a_{n-2,n-1}x_{n-2} + a_{n,n-1}x_n) \\
\dot{x}_n &= b_{n,1}x_1 + \ldots + b_{n,n}x_n + x_n(-a_{1,n}x_1 - a_{2,n}x_2 - \ldots. - a_{n-2,n}x_3 - a_{n-1,n}x_{n-1})
\end{aligned} \tag{23}$$

assuming that: $\sum_{k=1}^{n} b_{ki} = 0, \quad i = 1,2,\ldots.,n$ so that the integral

is preserved: $H(x_1,\ldots.,x_n) = \sum_{i=1}^{n} x_i = h = const.,$

We start again with the case n = 3 and demand that our system possess the P – property. Substituting Laurent expansions as before, we arrived at the following condition for the first order terms in the singular expansions:

$$N_3\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} -1 & A\alpha & B\alpha \\ -A\beta & -1 & D\beta \\ -B\gamma & -D\gamma & -1 \end{pmatrix}\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} -b_{11}\alpha - b_{12}\beta - b_{13}\gamma \\ -b_{21}\alpha - b_{22}\beta - b_{23}\gamma \\ -b_{31}\alpha - b_{32}\beta - b_{33}\gamma \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Since $P_1 + P_2 + P_3 = 0$ there will indeed be one free constant entering at this order, for all coefficients b_{ij} , i,j =1,2,3. More generally, since, for all n > 3, all columns of the matrices N_n add to 0, and the resonance condition is

$$r^n - r = (r+1)r^{n-2}(r-1) = 0$$

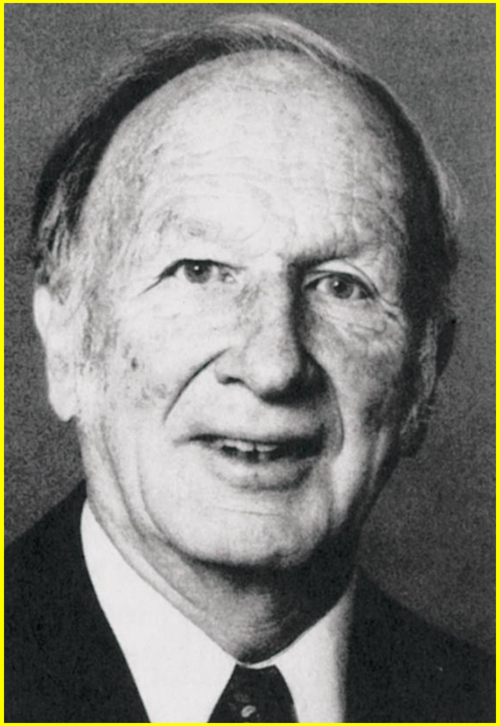
Since, $P_1 + P_2 + P_3 + ... + P_n = 0$, one free constant enters at r =1, n – 2 free constants enter at leading order (r = 0), and hence the P – property is satisfied for the nth order system (23) also.

There remains to prove the sufficiency for integrability:
I have solved completely by simple manipulations the equations (23) for n = 3 but not yet the general case of all n.

More generally, we know that integrable systems are:

a) Generally difficult to solve as they require sophisticated Mathematics
b) Their solutions have "simple" properties and do not show unexpected behavior as initial conditions or parameters change.
c) Thus, they occur rarely in the natural sciences, since most realistic systems are chaotic and unpredictable, and can only be solved numerically.

So, how "far" do we have to go from integrability to observe bifurcations leading to the "chaotic" behavior that we expect from non-integrable systems?



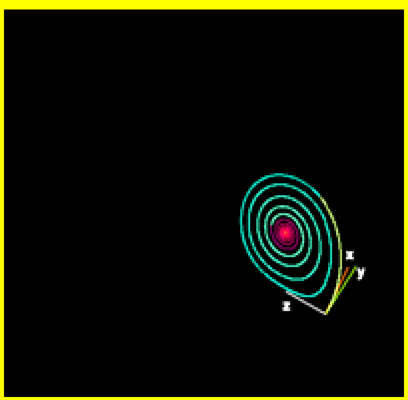
One of the first systems of nonlinear ODEs, (modeling the motion of warm air over a lake) was discovered by the American Meteorologist Edward Lorenz

$$\dot{x} = \sigma(y - x)$$

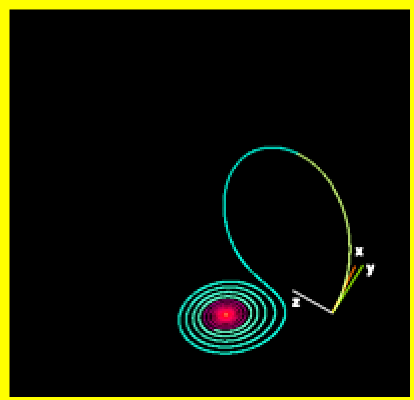
$$\dot{y} = \rho x - y - xz$$

$$\dot{z} = -bz + xy$$

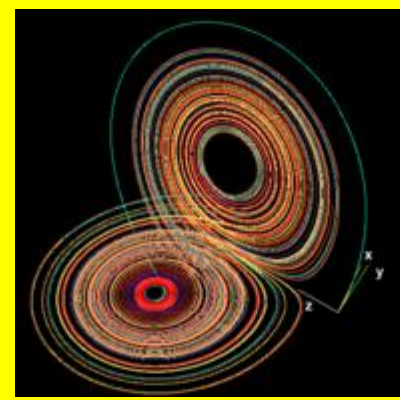
who solved them on his computer in 1963. This model has two non-zero fixed states for $\rho > 1$, which, as ρ increases become unstable, so for $\rho > 25$ solutions evolve chaotically from one to the other!



$\rho = 13$



$\rho = 15$



$\rho = 28$

3. Non-integrable perturbations of LVH systems

(with G. Kanellopoulos, University of Patras)

When integrable systems are perturbed, either by variation of their parameters, or adding new nonlinear terms they generally become non-integrable.

3.A So, we started with an n = 3 example, whose nonlinear terms are of the LVH type, but whose linear terms do not satisfy the integrability condition x + y +z = h =const., as follows:

$$\dot{x} = a_1 x + x(Ay + Bz)$$

$$\dot{y} = b_1 y + y(-Ax + Dz)$$

$$\dot{z} = a_2 x + b_2 y + cz + z(-Bx - Dy)$$

We studied fixed points of this dynamical system and varied the parameters to see if:

(a) They have stable and unstable manifolds that intersect, or

(b) They undergo interesting bifurcations, through which new solutions emerge.

The best we could do is find for c < 0 a stable spiral (focus) that appears through a degenerate Hopf bifurcation at (c =0), having a fairly large basin of attraction, see Figures 1 and 2 below.

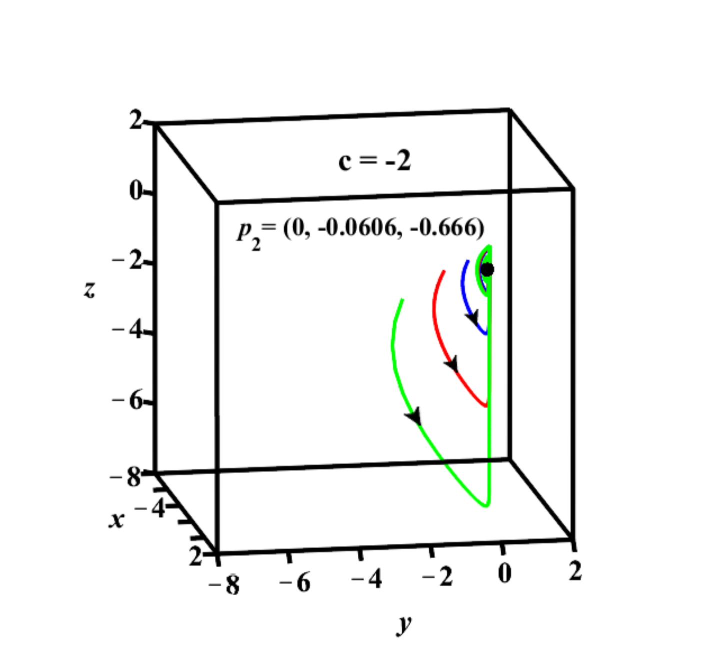


Fig. 1

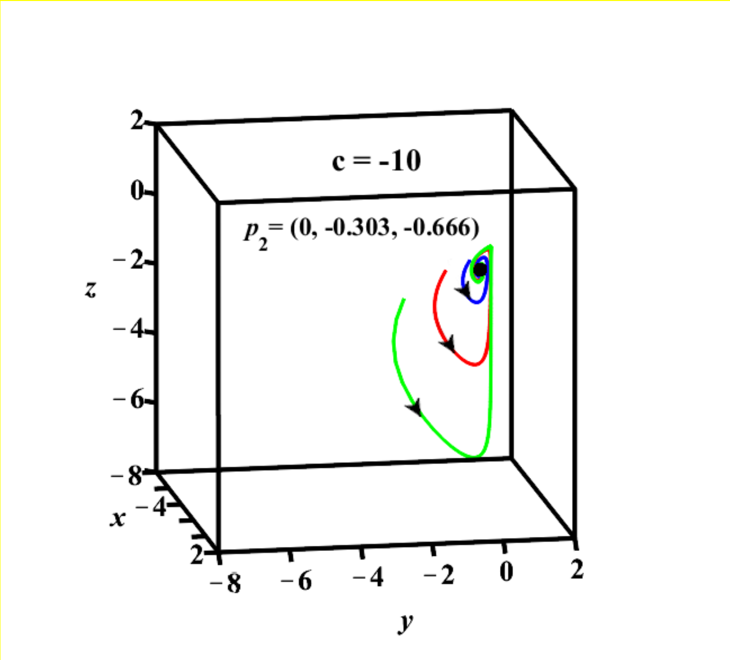


Fig. 2

Regarding their physical interpretation in population dynamics, if we change (x, y, z) to (-x, -y, -z), the stable fixed points and their basin of attraction shown in Figs. 1 and 2 below will occur for the system (25) in the x > 0, y > 0, z > 0 part of phase space.

3.B We also considered other n = 3 examples, whose nonlinear terms are of the LVH type, while a quadratic nonlinearity, modeling "deaths", is introduced as follows:

$$\begin{aligned}\dot{x} &= a_1 x + a_2 y + a_3 z + x(Ay + Bz) - \varepsilon_1 x^2 \\ \dot{y} &= b_1 x + b_2 y + b_3 z + y(-Ax + Dz) - \varepsilon_2 y^2 \\ \dot{z} &= c_1 x + c_2 y + c_3 z + z(-Bx - Dy) - \varepsilon_3 z^2\end{aligned} \tag{25}$$

We also kept the integrability condition B = D + A and varied A, B and D accordingly.

As before, the best we could do is find a "window" of A, B, D values for which there exists a stable focus with a sizable region of attraction.

Open question: How strongly must we perturb the equations of motion of the LV system to see interesting dynamics, or even chaos?

4. Connections Between the P-property and the CP formalism

Léon Brenig and his co-workers, in a number of papers [Phys. Rev. A 40, 4119 (1989), J. Math. Phys. Vol. 39 (5) 2929 – 2945 (1998), Phil. Trans. A, Proc. Royal Soc. 376, 20170384 (2018)] introduced the so – called Quasi-Polynomial (QP) formalism, in which a QP system is written in the form:

$$\frac{dx_i}{dt} = x_i \sum_{j=1}^{N} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \qquad i = 1,\dots,n \qquad (26)$$

where N and n are not necessarily equal. QP systems require two matrices: the Nxn matrix A and nxN matrix B, and employ the new variables:

$$u_j = \prod_{k=1}^{n} x_k^{B_{jk}} \qquad i = 1,\dots,n \qquad (27)$$

through which the equations of motion become

$$\frac{du_i}{dt} = u_i \sum_{j=1}^{N} M_{ij} u_j, \qquad i = 1,\dots,N$$

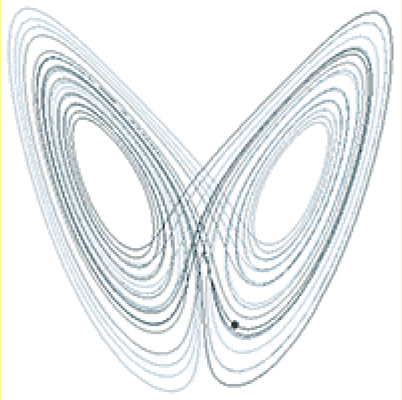
with M=BA, an NxN matrix.

A great benefit of the QP formalism is that it provides powerful criteria for:

a) Constructing invariants (first integrals) of the equations
b) Finding total or partial integrability conditions, through dimensional reductions.

Example 4.1: The Lorenz system

You all know the famous Lorenz system of first order ODEs:



$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= -x_2 + \rho x_1 - x_1 x_3 \\ \dot{x}_3 &= -b x_3 + x_1 x_2 \end{aligned} \qquad (28)$$

which exhibits chaos for wide ranges of its parameter values σ=10, ρ=28 and b=8/3. Are there any σ, ρ, β for which the equations are integrable?

Applying the QP formalism, we arrive at the following system of ODEs:

$$\frac{\dot{u}_1}{u_1} = -\sigma u_1 + \rho u_2 - u_3 + (\sigma - 1) \qquad (29)$$

$$\frac{\dot{u}_2}{u_2} = \sigma u_1 - \rho u_2 + u_3 + (1 - \sigma) \qquad (30)$$

$$\frac{\dot{u}_3}{u_3} = \sigma u_1 - \rho u_2 + u_3 + u_4 + (1 - \sigma - \beta) \qquad (31)$$

$$\frac{\dot{u}_4}{u_4} = \sigma u_1 + \rho u_2 - u_3 - u_4 + (\beta - \sigma - 1)C \qquad (32)$$

with the change of variables:

$$u_1 = x_1^{-1} x_2 \ , u_2 = x_1 x_2^{-1}, u_3 = x_1 x_2^{-1} x_3 \ , u_4 = x_1 x_2 x_3^{-1} \ , u_5 = 1$$

Combining the above relations (29) – (32) we find:

$$\frac{\dot{u}_1}{u_1} + \frac{\dot{u}_2}{u_2} = 0 \qquad \Rightarrow u_1 u_2 = const. = 1$$

And through simple manipulations, we are led to the expression

$$\frac{1}{u_2} D \exp\left(2\sigma \int \frac{dt}{u_2} - 2\sigma t \right) = u_4 E \exp\left(\int u_4 dt - \beta t \right) \quad (33)$$

where D and E are arbitrary constants. Now, observe that we can integrate equation (33) if β = 2σ and finally reduce the QP form of the Lorenz model to 2 equations:

$$\frac{\dot{u}_2}{u_2} = \frac{\sigma}{u_2} - \rho u_2 + u_3 + (1-\sigma) \quad (34)$$

$$\frac{\dot{u}_3}{u_3} = \frac{3\sigma}{u_2} - \rho u_2 + u_3 + (1-3\sigma) \quad (35)$$

It is known that the P-property leads to the following completely integrable cases:

1) σ =1, β = 2, ρ = 1/9 solved by Painlevé II transcendents.

2) σ =1/2, β =1, ρ =0 , which leads to elliptic functions,

Remarkably, the QP formalism finds the necessary Painlevé condition for integrability β = 2σ.

4.2 The May Leonard Equations of Competing Populations

In a celebrated paper, May R. H. and Leonard, W. J., SIAM J. Appl. Math. 29 (2), 243–253 (1975), the following (ML) 3 – dimensional system was studied:

$$\begin{aligned} \dot{x}_1 &= l_1 x_1 - x_1^2 - \alpha x_1 x_2 - \beta x_1 x_3 \\ \dot{x}_2 &= l_2 x_2 - x_2^2 - \beta x_1 x_2 - \alpha x_2 x_3 \\ \dot{x}_3 &= l_3 x_3 - x_3^2 - \alpha x_1 x_3 - \beta x_2 x_3 \end{aligned} \tag{36}$$

Using singularity analysis, we find that a necessary condition for the P – property is $\alpha + \beta = 2$, while the case $\alpha = \beta = 1$ is necessary and sufficient for the ML system to possess the P – property!

In the paper Figueiredo A., Rocha T.M. and Brenig L., J. Math. Phys. Vol. 39 (5) 2929 – 2945 (1998) the condition $\alpha + \beta = 2$ possesses the QP Invariant $I = (u_2\ u_3\ u_4)^{-1/3}$, while the case $\alpha = \beta = 1$ has the QP invariant

$$I = (u_2)^{\alpha}\, u_3^{[\alpha(l_2 - l_3) - l_3]/(l_3 - l_2)}\, u_4^{[\alpha(l_1 - l_2) - l_2]/(l_2 - l_3)}$$

for arbitrary values of l_1, l_2, l_3 !

4.3 A System of 3 – Wave Interactions

In the literature [Pikovskii A. S., and Rabinovich M.I., Math. Phys. Rev. vol. 2, Sov. Sci. Rev. C, Harwood (1981)], the system of ODEs

$$\begin{aligned} \dot{x}_1 &= \gamma x_1 + \delta x_2 + x_3 - 2x_2^2 \\ \dot{x}_2 &= \gamma x_2 - \delta x_1 + 2x_1 x_2 \\ \dot{x}_3 &= -2x_3 - 2x_1 x_3 \end{aligned} \qquad (37)$$

is said to represent 3 interacting, quasi – synchronous waves in a plasma with quadratic nonlinearities. In the notation of the QP formalism, this system gives rise to new variables:

$$u_1 = C = const., u_2 = x_1^{-1} x_2, u_3 = x_1^{-1} x_3, u_4 = x_1^{-1} x_2^2, u_5 = x_1 x_2^{-1}, u_6 = x_1$$

and a system of equations $\frac{du_i}{dt} = u_i \sum_{j=1}^{N} M_{ij} u_j, \qquad i = 1, \ldots, N$ with a matrix *M=BA, where*

$$B=\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A=\begin{pmatrix} \gamma & \delta & 1 & -2 & 0 & 0 \\ \gamma & 0 & 0 & 0 & -\delta & 2 \\ -2 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Here the equations of motion lead to the expressions:

$$u_2 u_5 = 1, \qquad u_4 = u_2^2 u_6$$

which reduce the above system to 3 equations:

$$\frac{\dot{u}_2}{u_2} = -\delta u_2 - u_3 + 2u_4 - \frac{\delta}{u_2} + 2\frac{u_4}{u_2^2} \qquad (38)$$

$$\frac{\dot{u}_3}{u_3} = -(\gamma+2) - \delta u_2 - u_3 + 2u_4 - 2\frac{u_4}{u_2^2} \qquad (39)$$

$$\frac{\dot{u}_4}{u_4} = \gamma - \delta u_2 - u_3 + 2u_4 - 2\frac{\delta}{u_2} + 4\frac{u_4}{u_2^2} \qquad (40)$$

Proceeding with further reduction of these equations, we arrive at the equation

$$3\frac{\dot{u}_2}{u_2} - \frac{\dot{u}_3}{u_3} - 2\frac{\dot{u}_4}{u_4} = \gamma + 2 + \frac{\delta}{u_2} \tag{41}$$

Observe now that if δ = 0, we can integrate directly this equation to find:

$$u_2^3 u_3^{-1} u_4^{-2} = De^{(\gamma+2)t} \tag{42}$$

where D is an integration constant. It is interesting to note here that δ = 0 is one of the conditions identified by the singularity analysis so that the system has the P-property [Bountis T., Grammaticos B., Ramani A., Physica 128A , 268, (1984)].

4.4 The Rikitake Dynamo Model of the Earth's Magnetic Field

In a paper by Rikitake T., Proc. Camb. Phys. Soc. Vol.54 , pp. 89 - 105 (1957), the following 3 – dimensional system was studied:

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + \beta x_2 + x_2 x_3 \\ \dot{x}_2 &= -\mu x_2 - \beta x_1 + x_3 x_1 \\ \dot{x}_3 &= \alpha - x_1 x_2 \end{aligned} \qquad (43)$$

with α, β, μ > 0 parameters representing angular momentum components. The change to QP variables is:

$$u_1 = C = const. = 1, u_2 = x_1^{-1} x_2, u_3 = x_1^{-1} x_2 x_3, u_4 = x_1 x_2^{-1},$$

$$u_5 = x_1 x_2^{-1} x_3, u_6 = x_1 x_2 x_3^{-1}, \ u_7 = x_3^{-1}$$

and leads to the equations:

$$
\begin{aligned}
\frac{\dot{u}_2}{u_2} &= -\beta u_2 - u_3 - \beta u_4 + u_5 \\
\frac{\dot{u}_3}{u_3} &= -\beta u_2 - u_3 - \beta u_4 + u_5 - u_6 + \alpha u_7 \\
\frac{\dot{u}_4}{u_4} &= \beta u_2 + u_3 + \beta u_4 - u_5 \\
\frac{\dot{u}_5}{u_5} &= \beta u_2 + u_3 + \beta u_4 - u_5 - u_6 + \alpha u_7 \\
\frac{\dot{u}_6}{u_6} &= -2\mu + \beta u_2 + u_3 - \beta u_4 + u_5 + u_6 - \alpha u_7 \\
\frac{\dot{u}_7}{u_7} &= u_6 - \alpha u_7
\end{aligned}
\tag{44}
$$

Amazing, isn't it? Now you start to think we are going the wrong way as we have to deal with 6 instead of 3 equations!

But don't worry!

Let us consider one of the two cases where the P – property indicates integrability: α = β = 0:

Using the above variable transformations, after some algebraic manipulations of the equations we find:

$$\frac{\dot{u}_3}{u_3} - \frac{\dot{u}_5}{u_5} = -2\frac{\dot{u}_4}{u_4} \;\Rightarrow\; u_3 u_4^2 = u_5 = \frac{u_3}{u_2^2}, \quad \frac{\dot{u}_4}{u_4} - \frac{\dot{u}_5}{u_5} = u_6 - \alpha u_7 = \frac{\dot{u}_7}{u_7} \Rightarrow\; u_4 = u_5 u_7 \;,$$

$$\frac{\dot{u}_2}{u_2} - \frac{\dot{u}_3}{u_3} = u_6 - \alpha u_7 = \frac{\dot{u}_7}{u_7} \Rightarrow u_2 = u_3 u_7 \;, \qquad u_2 u_4 = 1$$

Leading to the expression: $$\frac{\dot{u}_5}{u_5} + \frac{\dot{u}_6}{u_6} + \frac{2u_2 \dot{u}_2}{u_2^2 - 1} = -2\mu$$

which can be directly integrated to yield:

$$u_6\, u_5 (u_2^2 - 1) = C\, e^{-2\mu t} \qquad \Rightarrow \qquad x_2^2 - x_1^2 = C\, e^{-2\mu t}$$

All this leads to the single second order differential equation:

$$\ddot{u}_2 + \frac{2u_2\dot{u}_2^2}{1-u_2^2} = C\,e^{-2\mu t}$$

which can be directly integrated to obtain the solutions of the problem.

Now, there is also a second integrable case:

(B) μ = 0 and β = 0.

However, although this form has a simple integral in the original variables,

$$x_1^2 + x_2^2 + 2x_3^2 = const.$$

when transformed into canonical variables, this integral becomes a complicated polynomial that is difficult to extract by simply manipulating eq. (44).

CONCLUSIONS

1. Antisymmetric Lotka Volterra systems of n competing species without linear terms, possess the strong Painlevé property and are completely solvable by elementary functions.

2. Remarkably, they remain Painlevé integrable even when we add arbitrary linear terms to the equations (in a way that they preserve the Hamiltonian).

3. When we break their integrability by perturbing their parameters or adding new nonlinear terms, they continue to exhibit dynamically simple behavior!

4. It appears that the method of Quasi-Polynomial canonical forms can lead to Painlevé integrability conditions avoiding the long calculations of singularity analysis.

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To date, there are thousands of references on this topic and more results are obtained daily on its applications to PDEs and PΔEs.