Complicated behaviour for some Hénon–type maps

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The Hénon family

The well–known Hénon mapping reads as:

$$h : \mathbb{R}^2 \to \mathbb{R}^2, \;\; h(x, y) = (1 + y - ax^2, bx). \tag{1}$$

For $ab \neq 0$ it is conjugated to the mapping

$$h_s(x, y) = (y, -bx + a - y^2) \tag{2}$$

via mapping $L(x, y) = (a^{-1}y, a^{-1}bx)$.

That is, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{h_s} & \mathbb{R}^2 \\ L \downarrow & & \downarrow L \\ \mathbb{R}^2 & \xrightarrow{h} & \mathbb{R}^2 \end{array}$$

The generalized Hénon map of the plane reads as:

$$H : \mathbb{R}^2 \to \mathbb{R}^2, \; H(x, y) = (y, -\delta x + p(y)), \tag{3}$$

where $p(y)$ is a univariate polynomial.

Theorem

Any mapping belonging to the 2–dimensional affine Cremona group (that is, the group of polynomial mappings of the plane having polynomial inverse) is either a composition of mappings of the form above, or possesses trivial dynamics.

Proof.

Friedland S, Milnor J,, "Dynamical properties of plane polynomial automorphisms", Erg.Th.Dyn.Systems, 9, 67-99, 1989. □

Our model of interest

Take the so–called Klein-Gordon (KG) system of ODEs written in the form

$$\ddot{u}_n = -V'(u_n) + \alpha(u_{n+1} - 2u_n + u_{n-1}), \quad V(x) = \frac{1}{2}Kx^2 + \frac{1}{4}x^4, \tag{4}$$

$-\infty < n < \infty$.

Insert a Fourier series

$$u_n(t) = \sum_{k=-\infty}^{\infty} A_n(k) \exp(ik\omega_b t) \tag{5}$$

Set the terms proportional to the same exponential equal to zero to obtain the system of equations

$$-k^2\omega_b^2 A_n(k) = \alpha(A_{n+1}(k) - 2A_n(k) + A_{n-1}(k)) - KA_n(k) - \sum_{k_1,k_2,k_3} A_n(k_1)A_n(k_2)A_n(k_3), \quad k_1+k_2+k_3 = k, \qquad \forall k, n \in \mathbb{Z}\,, \quad (6)$$

We may substitute

$$u_n(t) = 2A_n(1)cos(\omega_b t)$$

in (4) and obtain a 2–dimensional map for the largest coefficient $A_n(1) = A_n(-1) = A_n$ as follows

$$-\omega_b^2 A_n = \alpha(A_{n+1} - 2A_n + A_{n-1}) - KA_n - 3A_n^3, \qquad (7)$$

If we now define $x_n = A_n$ and $y_n = A_{n-1}$ the above map takes the 2–dimensional form

$$\begin{aligned} x_{n+1} &= -y_n + Cx_n + \frac{3}{\alpha}x_n^3 \\ y_{n+1} &= x_n \end{aligned} \tag{8}$$

which is, of course, conjugate to a generalized Hénon map of the plane.

The 2–dimensional cubic map of interest

Let us focus now on the dynamics of

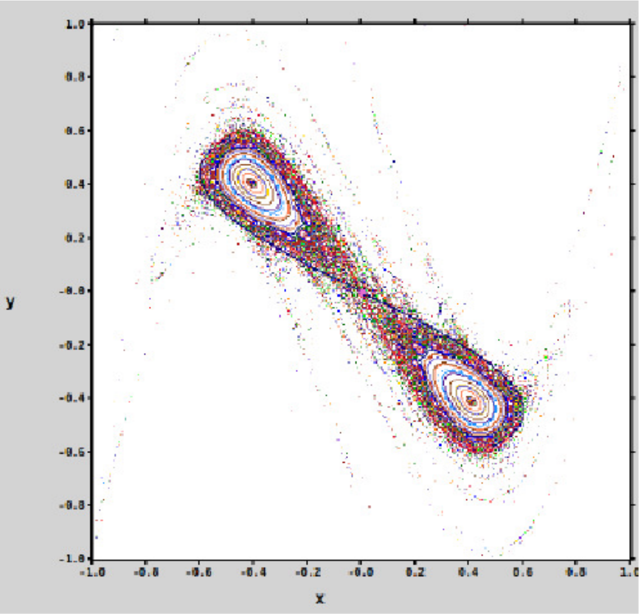
$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (y, -\delta x + cy + 3y^3).$$

It corresponds to the generalized Hénon map, with $p(y) = 3y^3 + cy$. Its inverse is:

$$f^{-1}(x, y) = (\tfrac{c}{\delta}x - \tfrac{1}{\delta}y + \tfrac{3}{\delta}x^3, x).$$

For $\delta = 1$, f is a symplectomorphism.

Denoting by σ how (x, y)maps to $(-x, -y)$ in the plane, the relation $f \circ \sigma = \sigma \circ f$ holds, meaning that f is invariant under this symmetry.

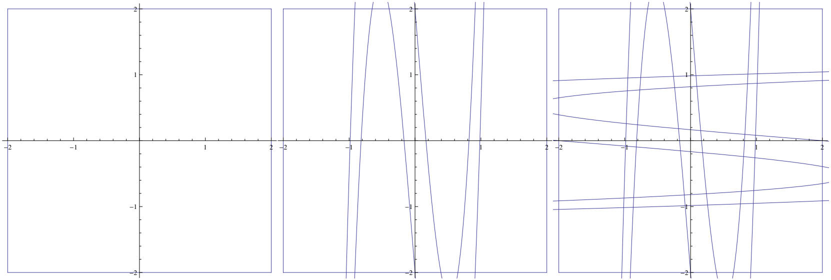

y
x

$\delta = 1$: Existence of a hyperbolic set

Definition: Let $g : M \to M$ be a diffeomorphism and Λ a compact and invariant by subset of M under this diffeomorphism. Λ is said to be a hyperbolic set for g if $\forall x \in \Lambda$ we have:

1. $T_x M = E_x^s \oplus E_x^u$,
2. $d_x g(E_x^{s,u}) = E_{g(x)}^{s,u}$,
3. $\|d_x g|_{E_x^s}\| < \lambda,\ \|d_x g^{-1}|_{E_x^u}\| < \mu^{-1}$, for $0 < \lambda < 1 < \mu$.

Proposition: Let f be the diffeomorphism defined above, with $c = -\frac{5}{2}$ and $\delta = 1$. There exists a hyperbolic set Λ on which f is topologically conjugate to the two–sided shift on three symbols. Moreover, Λ consist of all those points the orbits of which are bounded.



-2
-1
1
2

The Birkhoff–Smale Theorem

Let $f \in Diff^r(M)$, $r \leq 1$, M a smooth manifold.

Assume that $p \in M$ is a hyperbolic fixed point of f.

Let us also assume that $q \in W^u(p) \cap W^s(p)$, $p \neq q$ and that this intersection is transversal.

Theorem (Birkhoff–Smale)

Under the assumptions above, in a neighbourhood of p, there exists an invariant set Λ for f. Restricted on Λ, f is conjugated with the Bernoulli shift of two symbols.

Theorem above provides us a strategy for proving the existence of complicated behaviour in the phase space of a mapping.

One has "simply" to prove the existence of transverse homoclinic points.

But how to achieve that? Here, we focus our attention on the Parametrization Method.

Overview of the Parametrization Method

•Let $f : \mathbb{R}^n \to \mathbb{R}^n$ denote a C^∞ diffeomorphism, having a hyperbolic fixed point at $p \in \mathbb{R}^n$.

•Assume that E^s is the eigenspace of $d_p f : \mathbb{R}^n \to \mathbb{R}^n$, corresponding to eigenvalues that have norm less than one.

Theorem (Stable Manifold Theorem)

The stable set of p, $W^s(p) := \{x \in \mathbb{R}^n / lim_{n\to+\infty} f^n(x) = p\}$*, is a smooth, immersed submanifold of* $\mathbb{R}^n$*, tangent to* E^s *at p.*

PM builds on the following facts:

Under the assumptions imposed on the diffeomorphism f, there exists a C^∞ injective immersion $S : E^s \to \mathbb{R}^n$, such that:

(a) $S(p) = p$,

(b) the derivative of S at p is the inclusion map $E^s \hookrightarrow \mathbb{R}^n$, and

(c) $f \circ S = S \circ f^s$, where f^s stands for the restriction of $d_p f$ to E^s.

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{S} & \mathbb{R}^2 \\ f^s \downarrow & & \downarrow f \\ \mathbb{R}^2 & \xrightarrow{S} & \mathbb{R}^2 \end{array}$$

Back to our mapping

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (y, -\delta x + cy + 3y^3).$$

Fixed points: (0,0) and two symmetric ones, with coordinates

$$(-\frac{\sqrt{1-c+\delta}}{\sqrt{3}}, -\frac{\sqrt{1-c+\delta}}{\sqrt{3}}) \text{ and } (\frac{\sqrt{1-c+\delta}}{\sqrt{3}}, \frac{\sqrt{1-c+\delta}}{\sqrt{3}}).$$

Focusing at the fixed point of the origin, we first determine its eigenvalues:

$$\frac{1}{2}(c - \sqrt{c^2 - 4\delta}),\ \frac{1}{2}(c + \sqrt{c^2 - 4\delta}),$$

and proceed to study its invariant manifolds, beginning with the $\delta = 1$ case and continuing with $\delta < 1$.

Definition: Let $p \in W^s(0) \cap W^u(0)$, and denote by $S[0, p]$ the segment of $W^s(0)$ with endpoints 0 and p and by $U(0, p)$ the segment of $W^u(0)$ with endpoints 0 and p. The point p is called a primary intersection point if $S[0, p]$ intersects $U[0, p]$ only at the points p and 0.

The defining equation

Let $S_u : \mathbb{R} \to \mathbb{R}^2$ be the parametrization of the unstable manifold emanating from the origin:

$$S_u(t) = (\sum_{n=0}^{+\infty} a_n t^n, \sum_{n=0}^{+\infty} b_n t^n).$$

The defining equation of the unstable manifold becomes

$$f(S_u(t)) = S_u(\lambda_u t) \Rightarrow$$
$$(\sum_{n=0}^{+\infty} b_n t^n, -\delta \sum_{n=0}^{+\infty} a_n t^n + c \sum_{n=0}^{+\infty} b_n t^n + 3(\sum_{n=0}^{+\infty} b_n t^n)^3 =$$
$$(\sum_{n=0}^{+\infty} a_\lambda \lambda_u^n t^n, \sum_{n=0}^{+\infty} b_n \lambda_u^n t^n).$$

After equating terms of the same power of t, we get system:

$$\begin{aligned} -\lambda_u^n a_n + b_n &= 0 \\ -\delta a_n + (c + 9b_0^2 - \lambda_u^n) b_n &= s_{n-1}, \end{aligned} \tag{9}$$

where

$$s_{n-1} := -3(\sum_{j=1}^{n-1} b_0 b_{n-j} b_j + \sum_{i=1}^{n-1} \sum_{j=0}^{i} b_{n-i} b_{i-j} b_j).$$

Computations

In action:
set $a_0 = 0, \; b_0 = 0$ (coordinates of the fixed point),

set $a_1 = -0.447213595499579, \; b_1 = 0.894427190999915$
(components for the unstable eigenvector)

and evaluate that:

$a_2 = 0, \; b_2 = 0$

and continue to get:

$a_3 = 0,047702783519995504, \; b_3 = -0.38162226815996403.$ Of

course, after a while a computer might become handy...

1: $-0.447213595499958,\ 0.894427190999916$

2: $-0,\ -0$

3: $0.0477027835199955,\ -0.381622268159964$

4: $-0,\ -0$

5: $-0.00290759823359973,\ 0.093043143475191$2

6: $-0,\ -0$

7: $0.000114675022770544,\ -0.0146784029146296$

8: $-0,\ -0$

9: $-3.2356770268317e-06,\ 0.00165666663773783$

10: $-0,\ -0$

11: $7.01218341401969e-08,\ -0.000143609516319123$

12: $-0,\ -0$

13: $-1.22484683264699e-09,\ 1.00339452530442e-05$

14: $-0,\ -0$

15: $1.78267708241823e-11,\ -5.84147626366806e-07$

16: $-0,\ -0$

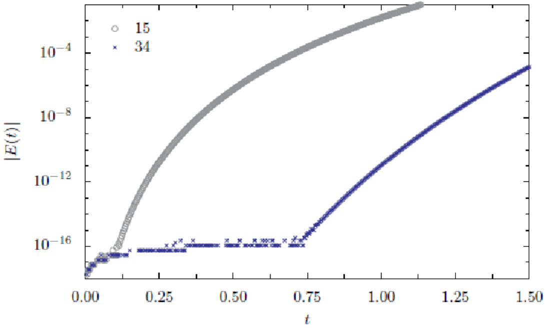
17: $-2.21378570778091e-13,\ 2.90165320290259e-08$

Range of validity

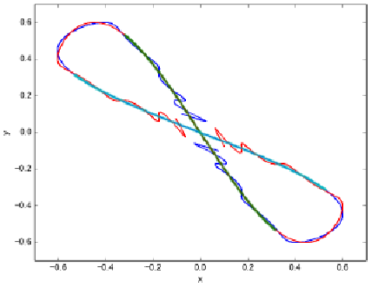
Ok but, where to stop?

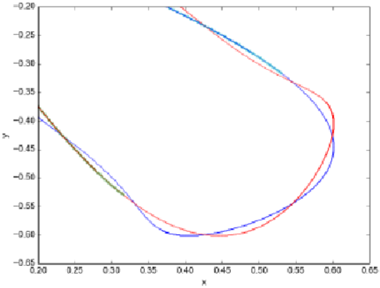
Definition: Let $\varepsilon > 0$. We shall say that $\tau > 0$ is an ε-radius of validity of the polynomial approximation $P_u(t)$ of $S_u(t)$ if

$$max_{t \in [-\tau, \tau]} ||f \circ P_u(t) - P_u(\lambda t)|| < \varepsilon.$$



Invariant manifolds





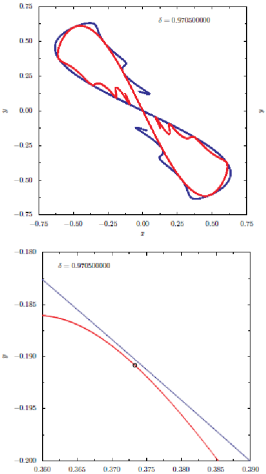
Evaluating the intersection points

Define the map:

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\varphi(t_1, t_2) = P_u(t_1) - P_s(t_2).$$

Non–degenerate zeros (t_1, t_2) of this map correspond to transverse homoclinic points of the mapping f.

So, we solved this equation, numerically, and found intesection points till the approximate value $\delta = 0.971398$.


$\delta = 0.970500000$
$\delta = 0.970500000$
$\delta = 0.970500000$
x
y

Two coupled cubic systems

If we wish to study breather interactions in two linearly coupled chains of one–dimensional Hamiltonian systems, the above analysis leads to the identification of homoclinic orbits of a 4–dimensional map of the form:

$$A_{n+1} - cA_n + \delta A_{n-1} = 3A_n^3 + b(A_n - B_n)$$
$$B_{n+1} - cB_n + \delta B_{n-1} = 3B_n^3 - b(A_n - B_n).$$

Setting now $A_{n-1} = x_1,\ A_n = y_1,\ B_{n-1} = x_2,\ B_n = y_2$, we define the mapping

$$f(x_1, y_1, x_2, y_2) =$$
$$(y_1, cy_1 - \delta x_1 + 3y_1^3 + b(y_1 - y_2), y_2, cy_2 - \delta x_2 + 3y_2^3 - b(y_1 - y_2)),$$

which is a diffeomorphism with inverse:

$$f^{-1}(x_1, y_1, x_2, y_2) =$$
$$(\tfrac{1}{\delta}((c+b)x_1 + 3x_1^3 - bx_2 - y_1), x_1, \tfrac{1}{\delta}((c+b)x_2 + 3x_2^3 - bx_1 - y_2), x_2).$$

The defining equation

Let $P : \mathbb{R}^2 \to \mathbb{R}^4$ be the parametrization of the unstable manifold of the origin

$$P(u,v) = \left(\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_1^{nm} u^n v^m, \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_2^{nm} u^n v^m\right), \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_3^{nm} u^n v^m, \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_4^{nm} u^n v^m)$$

The defining equation of the manifold becomes:

$$f \circ P(u,v) = P(\lambda_1 u, \lambda_2 v) \Rightarrow$$
$$\Rightarrow \Big(\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_2^{nm} u^n v^m, c\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_2^{nm} u^n v^m - \delta\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_1^{nm} u^n v^m + 3\Big(\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_2^{nm} u^n v^m\Big)^3 + \beta\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_4^{nm} u^n v^m,$$
$$\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_4^{nm} u^n v^m, c\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_4^{nm} u^n v^m - \delta\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_3^{nm} u^n v^m + 3\Big(\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_4^{nm} u^n v^m\Big)^3 - \beta\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_2^{nm} u^n v^m\Big) =$$
$$\Big(\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_1^{nm}\lambda_1^n\lambda_2^m u^n v^m, \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_2^{nm}\lambda_1^n\lambda_2^m u^n v^m, \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_3^{nm}\lambda_1^n\lambda_2^m u^n v^m, \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} a_4^{nm}\lambda_1^n\lambda_2^m u^n v^m\Big).$$

Equating terms of the same degree in this series, we arrive at the following system of equations, for the coefficients:

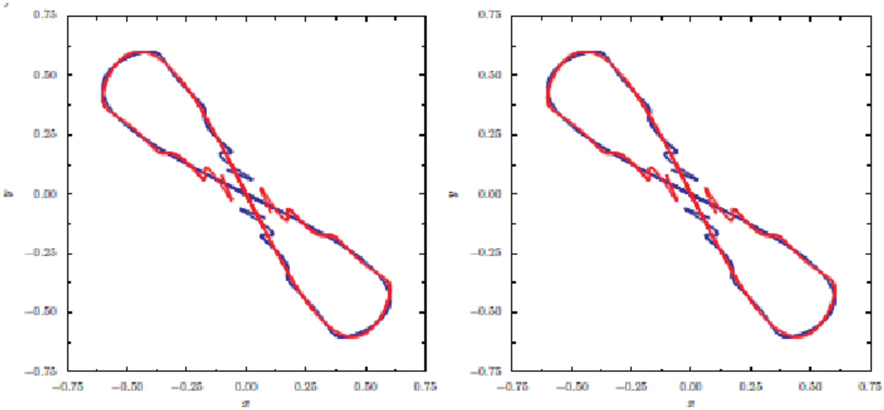
$$-\lambda_1^n \lambda_2^m a_1^{nm} + a_2^{nm} = 0$$

$$-\delta a_1^{nm} + (c + b - \lambda_1^n \lambda_2^m) a_2^{nm} - b a_4^{nm} = -3 \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{i=0}^{k} \sum_{j=0}^{l} a_2^{n-k,m-l} a_2^{k-i,l-j} a_2^{i,j}$$

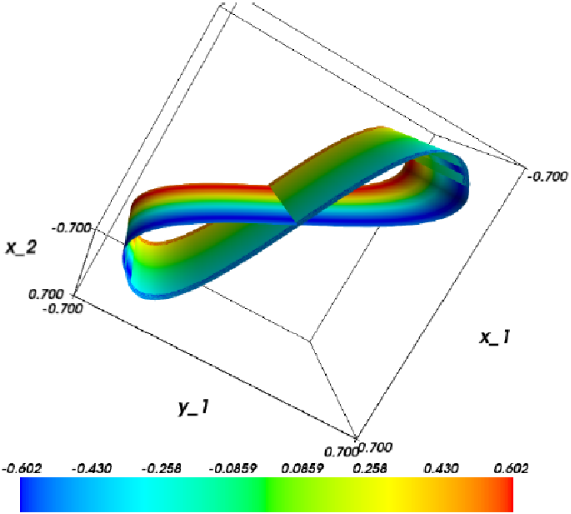
$$-\lambda_1^n \lambda_2^m a_3^{nm} + a_4^{nm} = 0$$

$$-b a_2^{nm} - \delta a_3^{nm} + (c + b - \lambda_1^n \lambda_2^m) a_4^{nm} = -3 \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{i=0}^{k} \sum_{j=0}^{l} a_4^{n-k,m-l} a_4^{k-i,l-j} a_4^{i,j}$$

b=0: the uncoupled case For $b = 0$ the maps in (x_1, y_1) and (x_2, y_2) are uncoupled. Thus in each degree of freedom we have the same 1D manifolds as before:

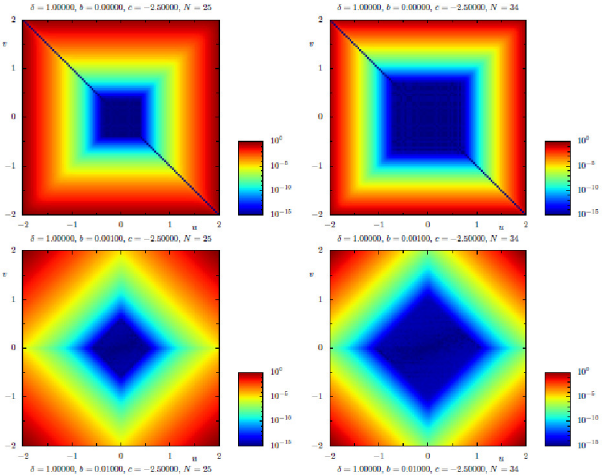


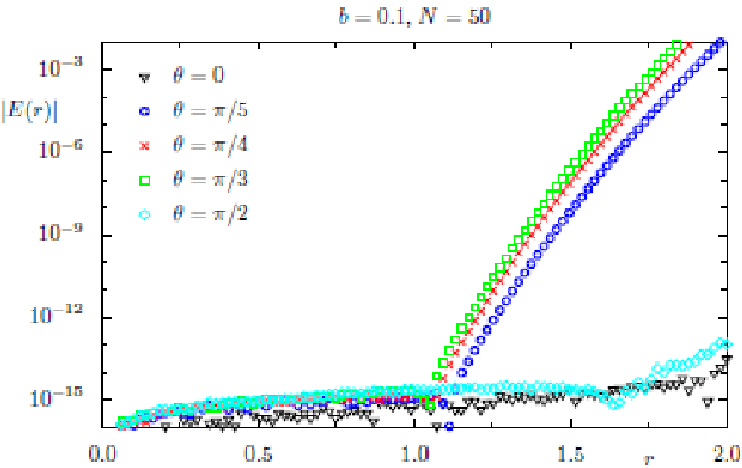
For the following plots $s, t \in [-1.75, 1.75]$ is used:



Validation issues...

Once again, we plot the error function:





$b = 0.1,\ N = 50$
10^{-3}
$|E(r)|$
10^{-6}
10^{-9}
10^{-12}
10^{-15}
$\theta = 0$
$\theta = \pi/5$
$\theta = \pi/4$
$\theta = \pi/3$
$\theta = \pi/2$
0.0
0.5
1.0
1.5
r
2.0

The final step

We feed these points to a numerical method of root finding, in order to get zeroes of the equation:

$$P_s(t_1, t_2) - P_u(t_3, t_4) = 0.$$

The results are much much better!

$$b = 0.1 \delta = 0.997$$

homoclinic point:

$$(-0.465214502071574,\ 0.498588601187636,$$
$$0.087251313184402,\ -0.089728305285299)$$

accuracy: $6.66134e - 16$.

Conclusions

- Studying a system of Klein-Gordon equations led us to the study of cubic Hénon maps.
- This map presents rich and complicated behaviour.
- We proved the existence of hyperbolic sets, in the symplectic case and explained their presence through the existence of transverse homoclinic points.
- We used Parametrization Method to locate such homoclinic points for various parameter values.
- Parametrization Method can be employed for studying invariant objects in various kinds of dynamical systems (odes, pdes, diffeomorphisms etc)

Some references

On the Parametrization Method:

1) Cabré X, Fontich E, and de la Llave R, "The parameterization method for invariant manifolds. III. Overview and applications", J. Differential Equations, 218(2):444-515, 2005.

2) Mireles James J D and Mischaikov K, "Rigorous A Posteriori Computation of (Un)Stable Manifolds and Connecting Orbits for Analytic Maps", SIAM J. Applied Dyn. Sys., 12(2):9571006, 2013.

On this lecture:

1) Anastassiou S, Bountis T and Bäcker A, "Homoclinic points of 2–D and 4–D maps via the parametrization method", Nonlinearity, 30, 3799-3820, 2017.