Complex Dynamics of the Laguerre iterative function

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1. Preliminaries
   - Dynamic systems
   - Iteration

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   - Iteration
2. Complex Analytic Dynamics
   - Complex Analysis
   - Julia and Fatou sets
   - The Mandelbrot set

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- At each point in the phase space there is a vector giving the direction of the infinitely small displacement of that point.
- We say then that all these vectors form a **dynamic system**.

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- **Complex dynamics** is the study of dynamic systems defined by iteration of functions on complex number spaces.
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- The iteration of complex mappings has a remarkable history, although it has undergone rapid development in the past thirty years.
- After a period of relative dormancy, the field was rejuvenated in 1980 thanks to some intriguing computer graphics images of Benoit Mandelbrot as well as major new mathematical advances due to Adrien Douady, John H. Hubbard, Dennis Sullivan and others.

(Discrete) dynamic systems

- For every $k \in \mathbb{N}$, we abbreviate as f^k the k-fold composition $f \circ f \circ \cdots \circ f$, where f^0 is the **identity function**. There should be no confusion with the ordinary power, which will be explicitly written as $[f(z)]^k$.

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- The **forward orbit** of a point $x \in X$ is the set
$$O^+(x) = \{f^n(x) : n \geq 0, f^0(x) = x\}.$$

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Complex variables

Definition

Let U be an open set in $\mathbb{C}$. A complex function $f : U \to \mathbb{C}$ is **differentiable** *or* **analytic** *at a point $a \in U$, if the* derivative

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad (z \in U)$$

exists. We call f **analytic on** *U, or say that $f : U \to \mathbb{C}$ is* **analytic**, *if f is analytic at each point of U.*

Periodic points

- A **periodic point** of period n of the transformation $f\colon X \to X$ is a point $x \in X$ such that $f^n(x) = x$ for some $n \in \mathbb{N}$.
- A periodic point of f of period 1 is called a **fixed point** of f.
- The orbit of a periodic point of f is called a **cycle** or **periodic orbit** of f.

The classification of periodic points

Suppose that $\zeta \in \mathbb{C}$ is a fixed point of an analytic function f and $\lambda = f'(\zeta)$. Then ζ is:

(a) *superattractive*, if $\lambda = 0$;

(b) *attractive*, if $0 < |\lambda| < 1$;

(c) *repulsive*, if $|\lambda| > 1$;

(d) *rationally indifferent*, if λ is a root of unity;

(e) *irrationally indifferent*, if $|\lambda| = 1$, but λ is not a root of unity.

Theorem

If $\deg(p) = n > 0$*, then* $z = \infty$ *is a superattractive fixed point of* p*.*

If f is invertible we can define the whole *orbit* of x as

$$O(x) = \{f^n(x) : n \in \mathbb{Z}, f^0(x) = x\}$$

and the *backward orbit* of x as

$$O^-(x) = \{f^{-n}(x) : n \geq 0, f^0(x) = x\}.$$

Definition

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a function. If a is a (super)attractive periodic point of f, we write

$$A(a) = \{z \in \overline{\mathbb{C}} : \lim_{n \to \infty} f^n(z) = a\}$$

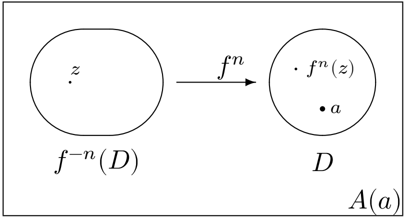
for the **attractive basin** *or* **stable set** *or* **basin of attraction** *of a, i.e. the basin of attraction is the set of points which approximate a given (super)attractive periodic orbit.*

Remarks

Obviously, $O^-(a) \subset A(a)$ and $A(a) \neq \emptyset$, because $a \in A(a)$. We observe that if a is an attractive fixed point of f, then $f^k(z) \in A(a)$ for some $k \in \mathbb{N}$ implies $z \in A(a)$.

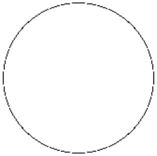
Theorem

A basin of attraction is an open set.



Example

Let $f(z) = z^2$; so that $f^n(z) = z^{2^n}$ for $z \in \mathbb{C}$. This is the simplest function of type $f(z) = z^2 + c$ and it will prove instructive to see how the geometry changes, sometimes dramatically, as c is altered. In our case $c = 0$ the fixed points are a repeller $z = 1$, since $f'(1) = 2$ and a superattractor $z = 0$.



The Julia set of z^2.

The dynamic dichotomy

About 1918–1920, the French mathematicians P. Fatou and G. Julia, developed, independently of each other, the theory of rational iteration, their main tool being Montel's Normality Criterion.

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They discovered the dichotomy of the Riemann sphere into the sets now bearing their names.

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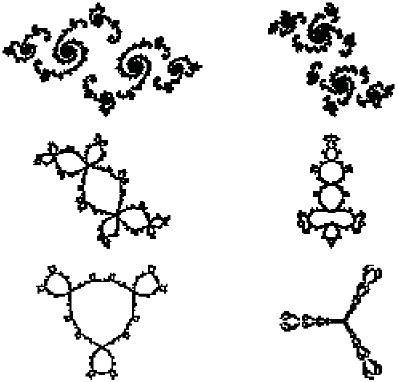
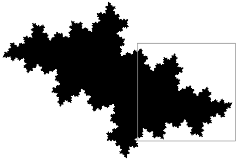
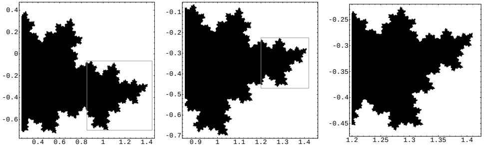
Several Julia sets



Quasi self-similarity

A looser form of self-similarity; the fractal appears approximately (but not exactly) identical at different scales.





$$c = -0.5 + 0.5i$$

Möbius transformations

A transformation $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the form

$$R(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ with

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 \Leftrightarrow ad - bc \neq 0 \Leftrightarrow ad \neq bc,$$

is called a **bilinear**, **linear fractional** or **Möbius transformation.**

If $c \neq 0$, then $R(-d/c) = \infty$ and $R(\infty) = a/c$. If $c = 0$, then $R(\infty) = \infty$. Its inverse is

$$R^{-1}(z) = \frac{dz-b}{-cz+a},$$

if $c \neq 0$.

Converting quadratics

A quadratic polynomial

$$q(z) = az^2 + 2bz + d \quad (a \neq 0),$$

where $a, b, d \in \mathbb{C}$, may be simplified by an affine change of coordinates

$$w = \Phi(z) = az + b \quad (a \neq 0)$$

to the form

$$p_c(z) = z^2 + c,$$

where $c = ad - b^2 + b$. We show how the process works with a **commutative diagram**

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{q} & \overline{\mathbb{C}} \\ \Phi^{-1} \uparrow & & \downarrow \Phi \\ \overline{\mathbb{C}} & \xrightarrow{p_c} & \overline{\mathbb{C}} \end{array}$$

where Φ is a Möbius transformation, to mean that all compositions taking us from one given point to another are equal.

Rational functions

- A **rational map** $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is of the form $R = P/Q$, where P and Q are polynomials without common factors and so without common roots.
- The **degree** of R is defined by
$$\deg(R) = \max\{\deg(P), \deg(Q)\}.$$
- How is $R(\infty)$ defined?

Critical points

- In order to understand the dynamics of all complex quadratic polynomials, it is enough to study the class of quadratic polynomials of the form $z \mapsto z^2 + c, c \in \mathbb{C}$.

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is 0, with c as its critical value (parameter).
- The forward orbits of the critical points of a rational map determine the general features of the global dynamics of the map

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- These different sets containing the convergent points are separated from some other set which usually has a nonintegral dimension.

Julia sets of quadratics

In what follows, $p_c(z) = z^2 + c$, where $z, c \in \mathbb{C}$ and $J(p_c) = J_c$.

Theorem

p_c has at most one finite attractive fixed point or attractive cycle.

Theorem

If $\lim_{n\to\infty} p_c^n(0) \neq \infty$, then J_c is connected.

Theorem

If $\lim_{n\to\infty} p_c^n(0) = \infty$, then J_c is totally disconnected.

The dichotomy between the c parameter values of p_c for which either convergence or divergence is implied, was studied by Mandelbrot.

Definition

$$\begin{aligned}\mathcal{M} &= \{c \in \mathbb{C} : \lim_{n\to\infty} p_c^n(0) \neq \infty\} \\ &= \{c \in \mathbb{C} : \{p_c^n(0)\}_{n=1}^{\infty} \textit{ is bounded}\}.\end{aligned}$$

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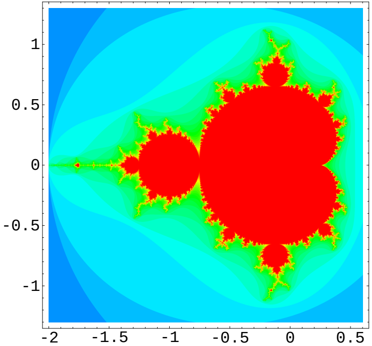

Douady–Hubbard

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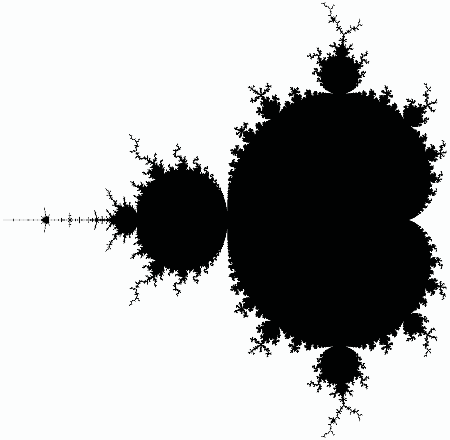

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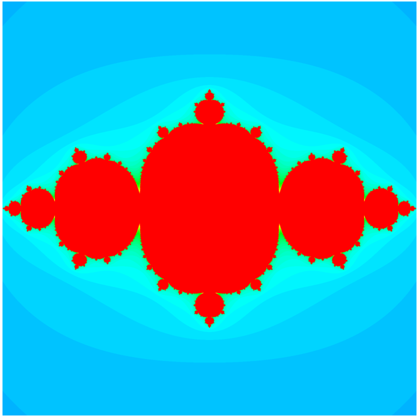
Definition

The filled-in Julia set, $K(f)$, *of* f *is the set of points whose orbits do not tend to infinity, i.e.,*

$$\begin{aligned} K(f) &= \{z \in \overline{\mathbb{C}} : \lim_{n\to\infty} f^n(z) \neq \infty\} \\ &= \{z \in \overline{\mathbb{C}} : \{|f^n(z)|\}_{n=0}^{\infty} \textit{ is bounded}\}. \end{aligned}$$

San Marco dragon



The fractal $J(-3/4, 0)$, where J is the Julia set. It slightly resembles the Mandelbrot set.

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3. Iterative methods
   - Newton-Raphson's method
   - Laguerre's method

Finding roots

Using an iterative method to find the roots of a polynomial equation

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0, \quad a_d \neq 0$$

where $p\colon \mathbb{C} \to \mathbb{C}$ is a complex polynomial of a complex variable, is identical to computing individual orbits of the dynamic system generated by the method.

Questions

- What is the open set of all initial values for which iteration sequence converges to a given root?
- For what initial point on the extended complex plane will the sequence not converge at all?
- Is it possible that iteration sequence converges to points or cycles other than the desired roots?

Definition

The *Newton-Raphson's iterative method* for finding the complex roots of the equation $f(z) = 0$, where f is an arbitrary function, is

$$N(z) = z - \frac{f(z)}{f'(z)}$$

for $f'(z) \neq 0$.

Remarks

- The roots of $p(z)$ correspond to the finite fixed points of $N_p(z)$.
- The point at infinity is a fixed point of N_p and since $N_p'(\infty) = d/(d-1)$, it is repulsive.
- The derivative of N_p is
$$N_p'(z) = \frac{p(z)p''(z)}{[p'(z)]^2},$$
and therefore, the simple roots of $p(z)$ are superattractive fixed points of $N_p(z)$.
- In a neighbourhood of its superattractive fixed points, the algorithm is locally conjugate to $z \mapsto z^k$ for some $k > 1$. Thus, local convergence is very rapid.
- For a generic polynomial of degree d, the Newton's map is a rational map of degree d. When the polynomial has multiple roots, $\deg(N_p) < d$.

Multiple roots of p

- They are attractive fixed points, but not superattractive.
- In fact,

$$N_p'(z)\big|_\xi = \frac{m-1}{m} = 1 - \frac{1}{m}$$

  where ξ is the root of multiplicity m.
- The **rate of attraction** (speed of convergence) is linear and the algorithm is not very effective in this case.

Newton's method for quadratics

- We have seen that any quadratic polynomial may be reduced to the form $p_c(z) = z^2 - c$ (the minus sign is convenient here) by an affine change of coordinates.
- We determine for Newton's iterative method the attractive basin of each complex root $\pm\sqrt{c}$.
- Here we iterate with the rational function

$$N(z) = z - \frac{p_c(z)}{p_c'(z)} = \frac{z^2 + c}{2z}.$$

- Notice that, apart from ∞, the fixed points of $N(z)$ are the roots $\pm\sqrt{c}$, as we would expect.
- Let us use a Möbius transformation to map the dynamics to an equivalent system which is much easier to handle.
- It is a good idea to send $\pm\sqrt{c}$ to the points $\infty, 0$, respectively, which is accomplished for example by

$$w = \Phi(z) = \frac{z+\sqrt{c}}{z-\sqrt{c}}.$$

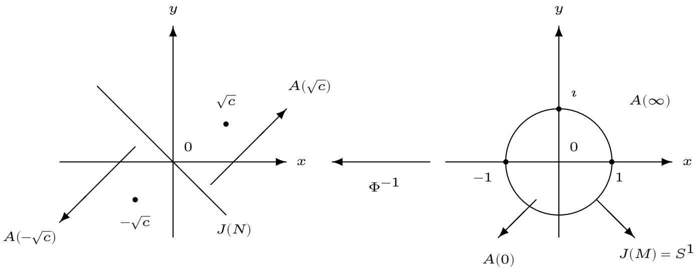
The inverse is

$$\Phi^{-1}(w) = \frac{-w\sqrt{c}-\sqrt{c}}{-w+1} = \sqrt{c}\,\frac{w+1}{w-1}.$$

If $N(z)$ becomes $M(w)$, then

$$\begin{aligned} M(w) &= \Phi \circ N \circ \Phi^{-1}(w) = \Phi \circ N\left(\sqrt{c}\,\frac{w+1}{w-1}\right) \\ &= \Phi\left(\sqrt{c}\,\frac{w^2+1}{w^2-1}\right) = \frac{(w^2+1)/(w^2-1)+1}{(w^2+1)/(w^2-1)-1} \\ &= w^2. \end{aligned}$$

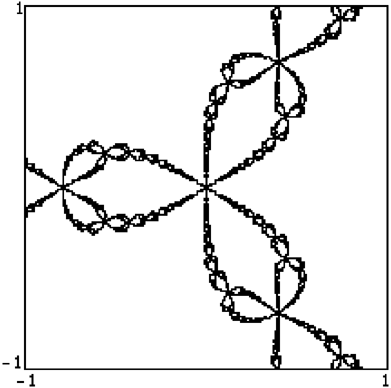
We already know that $J(M) = S^1$, with interior $A(0)$ and exterior $A(\infty)$. Tranforming back to the $z-$plane we obtain:
$\Phi^{-1}(1) = \infty$, therefore $\Phi^{-1}(S^1)$ is a straight line.
$\Phi^{-1}(-1) = 0$, therefore this line contains the origin.
$\Phi^{-1}(\imath) = -\imath\sqrt{c}$, a complex number at right angles to $\sqrt{c}$.



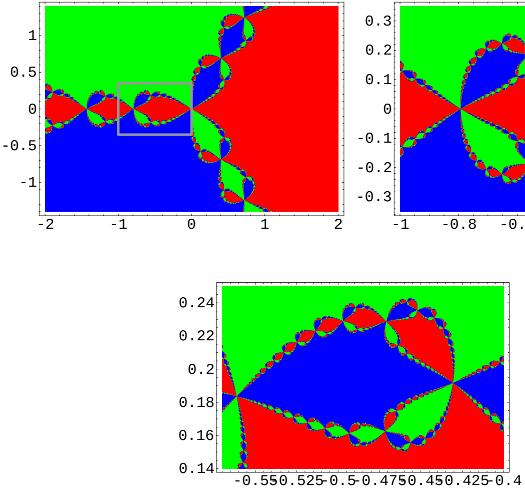
Thus the Julia set of N, $J(N) = \Phi^{-1}(S^1)$, is the perpendicular bisector of the line segment joining $-\sqrt{c}$ to $\sqrt{c}$. Further, we conclude that $A(\pm\sqrt{c})$ are the half-planes on opposing sides of $J(N)$.

Complexity: Fractal basin boundaries

In the Julia set of Newton's function for the iterative solution of $z^3 = 1$, every point is 3-cornered. That is, every point is in the boundary of each root's basin of attraction



The Julia set of the N_{p_3} method



The convergence domain of $z^3 - 1 = 0$ with Newton's method and $f'(z) \neq 0$.
With progressive enlargements we again discern the phenomenon of self-similarity.

1. Preliminaries
   - Dynamic systems
   - Iteration
2. Complex Analytic Dynamics
   - Complex Analysis
   - Julia and Fatou sets
   - The Mandelbrot set
3. Iterative methods
   - Newton-Raphson's method
   - Laguerre's method

Laguerre iterative function

$$L(z) = z - \frac{\nu f(z)}{f'(z) + \{(\nu-1)^2[f'(z)]^2 - \nu(\nu-1)f(z)f''(z)\}^{1/2}}, \quad (1)$$

where the argument of the root is to be chosen to differ by less than $\pi/2$ from the argument of $(\nu-1)f'(z)$. Eq. 1 can be written in equivalent form as

$$L(z) = z - \frac{\nu[f(z)/f'(z)]}{1 + \{(\nu-1)^2 - \nu(\nu-1)[f(z)f''(z)/(f'(z))^2]\}^{1/2}}. \quad (2)$$

The iteration (2) for $\nu = 2$ becomes

$$L_2(z) = z - \frac{2[f(z)/f'(z)]}{1 + \{1 - 2[f(z)f''(z)/(f'(z))^2]\}^{1/2}}. \quad (3)$$

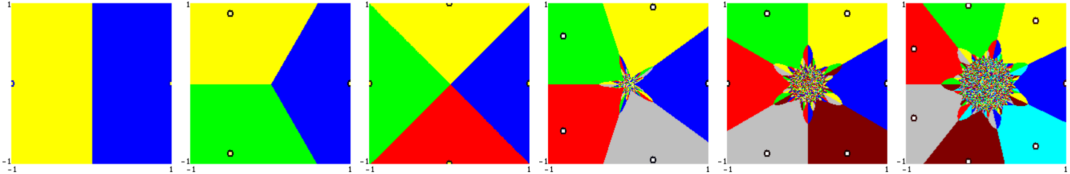
The convergence is cubic to a simple root and linear to a multiple

Basins of attraction for the roots of $z^n = 1$

$$L(z) = z\frac{z^{-n/2} + (n-1)}{z^{n/2} + (n-1)}.$$

For even integer n it is rational, for odd n it is algebraic. The fixed point condition $L(z) = z$ implies that

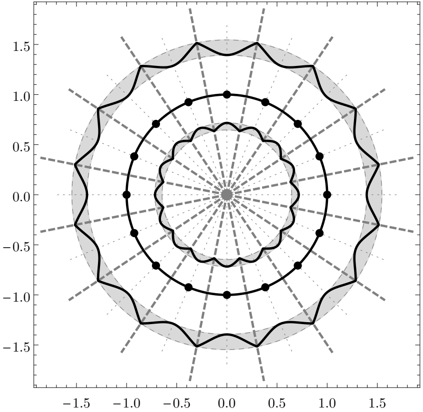
- $z^n - 1 = 0$ or
- $z = 0$, repulsive periodic point of period 2



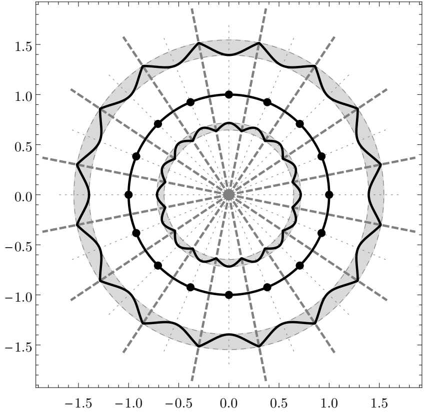
Remarks

- When $n = 2$, convergence occurs in one iteration for any starting point $z_0 \in \mathbb{C}$. If $\Re(z_0) > 0$, or if $\Re(z_0) = 0$ and $\Im(z_0) \geq 0$, then $L(z_0) = 1$; otherwise $L(z_0) = -1$.
- When $n \geq 3$, $z = 0$ is a repelling periodic point of period 2, because $L(0) = \infty$ and $L(\infty) = 0$. For any other starting point convergence to the roots of f_n occurs for $n = 3$ and $n = 4$.
- The situation is much more interesting for $n \geq 5$.
- In this case, the $\{0, \infty\}$ two-cycle becomes attracting and the Lebesgue measure of the roots' basins of attraction approaches 0 as $n \to \infty$.
- More specifically, these basins are subsets of an annulus whose radii r_1 and r_2 satisfy $0 < r_1 < 1 < r_2 = 1/r_1 < (n-1)^{2/(n-4)}$, so both radii converge to 1 as $n \to \infty$.

- The case with $n = 16$ is shown with the 16 roots shown as dots along the unit circle, and two grey annuli that vaguely outline the boundary of the union of the basins of attraction of the roots



- The case with $n = 16$ is shown with the 16 roots shown as dots along the unit circle, and two grey annuli that vaguely outline the boundary of the union of the basins of attraction of the roots
- Laguerre iteration converges to the roots for any starting point between the two grey annuli, but it approaches the $\{0, \infty\}$ two-cycle for any starting point inside the smaller grey annulus or outside the larger grey annulus

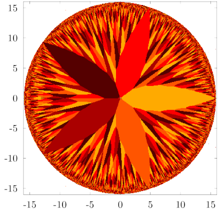


Symmetry effects

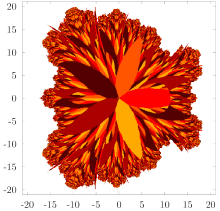
- Theoretical considerations and the numerical results show that the Laguerre iteration has similar dynamics as for the case with $f_n(z) = z^n - 1$ in the previous section: in both cases the union of the basins of attraction to the roots is a subset of an annulus centered at the origin and $\{0, \infty\}$ is an attracting two-cycle.
- Let us consider the polynomial $p_r(z) = (z - r)(z^4 + z^3 + z^2 + z + 1)$ in which r is viewed as a perturbation of the real root $z^* = 1$ of $f_5(z) = z^5 - 1$.
- We present the results of small perturbations with $r \in \mathbb{R}$ and the consequent changes in the roots' basins of attraction.

Example

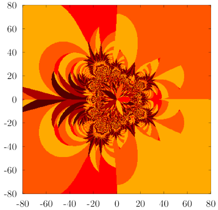
We show the original basin with $r = 1$ and two small perturbations: one with $r = 1.00060$ and one with $r = 1.00064$. Notice how sensitive the results are to these two perturbations. In the middle image the union of the basins of attraction has slowly grown and slightly changed its shape. When changing r from 1.00060 to 1.00064, there is a sudden change and as far as we could computationally check, the basins visibly cover the complex plane.



(a)



(b)



(c)

Conclusions

- There are polynomials for which convergence to the roots, at least when exact arithmetic is used, will not take place due to the existence of attracting cycles
- When the symmetry of the roots of unity is slightly perturbed, convergence again seems to take place from much or all of the complex plane
- Unlike with rational iteration maps, under Laguerre's iteration the boundaries of the individual roots' basins of attraction do not correspond to the Julia sets of the polynomials.