Χάος γύρω από μιγαδικά ασταθείς περιοδικές τροχιές σε 3D Χαμιλτονιανά συστήματα

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ΚΕΑΕΜ, Ακαδημίας Αθηνών

In 2D systems there is no Complex Instability

Equations of motion are derived from the Hamiltonian

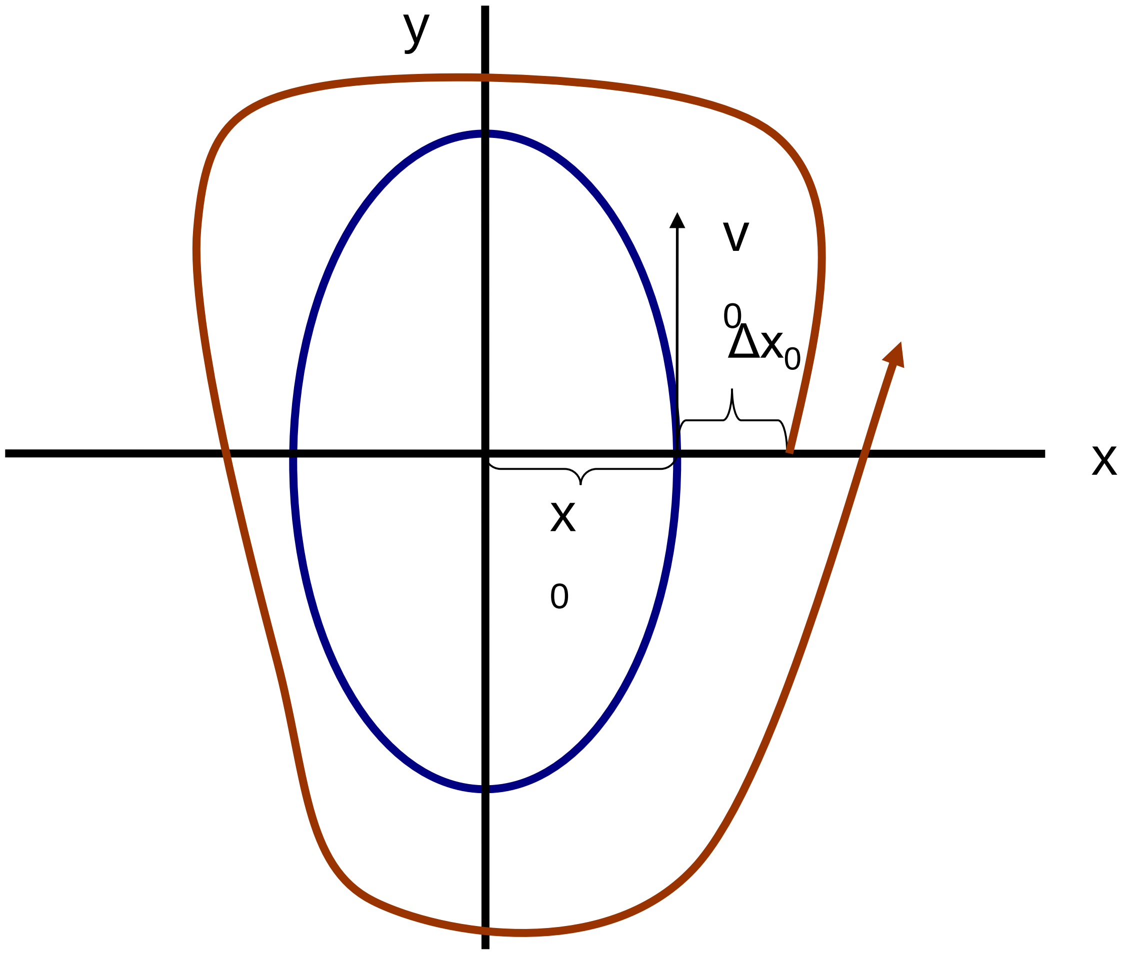
$$H \equiv \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \Phi(x, y) - \frac{1}{2}\Omega_s^2(x^2 + y^2) = E_J \qquad (4)$$

where (x, y) are the coordinates in a Cartesian frame of reference corotating with the spiral with angular velocity Ω_s. $\Phi(x, y)$ is the potential in Cartesian coordinates, E_J is the numerical value of the Jacobian integral and dots denote time derivatives.

Effective potential
takes care of fictitious forces

E_J, the Jacobi integral, is the rotating-frame analog of the total energy

Ευστάθεια περιοδικών τροχιών (Hénon 1965)



6.1 Ευστάθεια κατά Henón

Η ευστάθεια υπολογίζεται με τη μέθοδο του *Henón* (1965). Ξεκινώντας από τον τετραδιάστατο χώρο των φάσεων $(x, y, \dot{x}, \dot{y})$, θεωρούμε τις διαδοχικές τομές μιας τροχιάς με τον άξονα $y = 0$, κατά τη διεύθυνση των αυξανόμενων y ($\dot{y} > 0$). Από τη Χαμιλτονιανή $H = H(x, 0, \dot{x}, \dot{y}) = h$ μπορούμε να λύσουμε ως προς $\dot{y}$ και έτσι ο χώρος των φάσεων περιορίζεται σε δύο αρχικές συνθήκες $(x, \dot{x})$.

Δύο διαδοχικά σημεία τομής στον άξονα $y = 0$ συνδέονται με έναν μετασχηματισμό $\mathbb{R}^2 \to \mathbb{R}^2$. Για την περίπτωση της περιοδικής τροχιάς έχουμε:

$$x_0 = g_1(x_0, \dot{x}_0)$$
$$\dot{x}_0 = g_2(x_0, \dot{x}_0)$$

Εισάγωντας μια μικρή διαταραχή στις αρχικές συνθήκες παίρνουμε μια τροχιά γειτονική της αρχικής $(x_0 + \Delta x_0, \dot{x}_0 + \Delta \dot{x}_0)$. Οι αρχικές και οι τελικές συνθήκες συνδέονται πάλι μέσω του μετασχηματισμού και έχουμε:

$$x_0 + \Delta x_1 = g_1(x_0 + \Delta x_0, \dot{x}_0 + \Delta \dot{x}_0)$$
$$\dot{x}_0 + \Delta \dot{x}_1 = g_2(x_0 + \Delta x_0, \dot{x}_0 + \Delta \dot{x}_0)$$

Αναπτύσσοντας κατά Taylor και κρατώντας όρους μέχρι πρώτης τάξης έχουμε:

$$\Delta x_1 = \frac{\partial g_1}{\partial x}\Delta x_0 + \frac{\partial g_1}{\partial \dot{x}}\Delta \dot{x}_0$$
$$\Delta \dot{x}_1 = \frac{\partial g_2}{\partial x}\Delta x_0 + \frac{\partial g_2}{\partial \dot{x}}\Delta \dot{x}_0$$

ή αναλυτικά:

$$\Delta x_1 = a\Delta x_0 + b\Delta \dot{x}_0$$
$$\Delta \dot{x}_1 = c\Delta x_0 + d\Delta \dot{x}_0$$

όπου $a = \frac{\partial g_1}{\partial x}$, $b = \frac{\partial g_1}{\partial \dot{x}}$, $c = \frac{\partial g_2}{\partial x}$, $d = \frac{\partial g_2}{\partial \dot{x}}$

Δεδομένου ότι ο μετασχηματισμός διατηρεί τα εμβαδά, έχουμε

$$ad - bc = 1 \tag{3}$$

Επομένως $\overrightarrow{k}_1 = \mathbf{A}\overrightarrow{k}_0$, όπου $\overrightarrow{k}_1$ είναι το διάνυσμα $(\Delta x_1, \Delta \dot{x}_1)$ και $\overrightarrow{k}_0$ το διάνυσμα $(\Delta x_0, \Delta \dot{x}_0)$.

Εάν $\{\overrightarrow{\delta}_1, \overrightarrow{\delta}_2\}$ η βάση των ιδιοδιανυσμάτων, μπορούμε να γράψουμε:

$$\overrightarrow{k}_0 = A_1\overrightarrow{\delta}_1 + A_2\overrightarrow{\delta}_2$$
$$\overrightarrow{k}_1 = A_1\lambda_1\overrightarrow{\delta}_1 + A_2\lambda_2\overrightarrow{\delta}_2$$

όπου λ_1 και λ_2 οι ιδιοτιμές της Ιακωβιανής **A**. Η χαρακτηριστική εξίσωση του όπου λ_1 και λ_2 οι ιδιοτιμές της Ιακωβιανής **A**. Η χαρακτηριστική εξίσωση του πίνακα **A**, λόγω της σχέσης (3) είναι

$$\lambda^2 - (a + d)\lambda + 1 = 0$$

Στην περίπτωση που έχουμε $|a + d| < 2$, έχουμε δύο ρίζες μιγαδικές συζυγείς. Σε αυτήν την περίπτωση $|\lambda_1| = |\lambda_2| = 1$, και η τροχιά χαρακτηρίζεται ευσταθής.

Εάν έχουμε $|a + d| > 2$ τότε έχουμε δύο πραγματικές ρίζες, με $\lambda_1\lambda_2 = 1$ και η τροχιά χαρακτηρίζεται ασταθής. Ως δείκτης ευστάθειας ορίζεται ως εκ τούτου η παράμετρος

$$\alpha = \frac{1}{2}(a + d)$$

Για $|a| < 1$ η περιοδική τροχιά είναι ευσταθής, ενώ για $|a| > 1$ είναι ασταθής. Το διάγραμμα που δίνει τον δείκτη ευστάθειας α ως συνάρτηση της ενέργειας ονομάζεται διάγραμμα ευστάθειας.

Henon's index

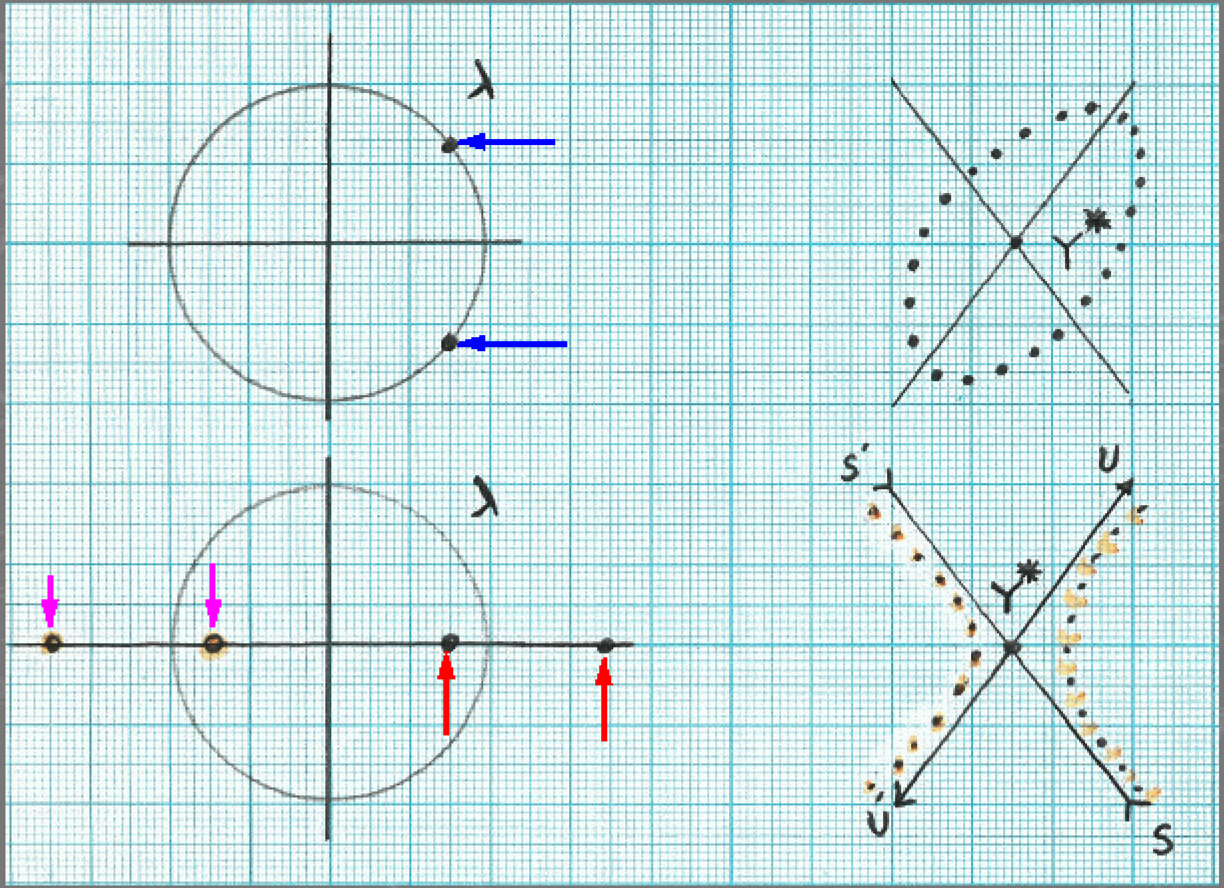
Characteristic equation:

$$\lambda^2-(a+d)\lambda+1=0$$

$$\alpha=1/2(a+d)$$

$|\alpha|<1$ STABLE

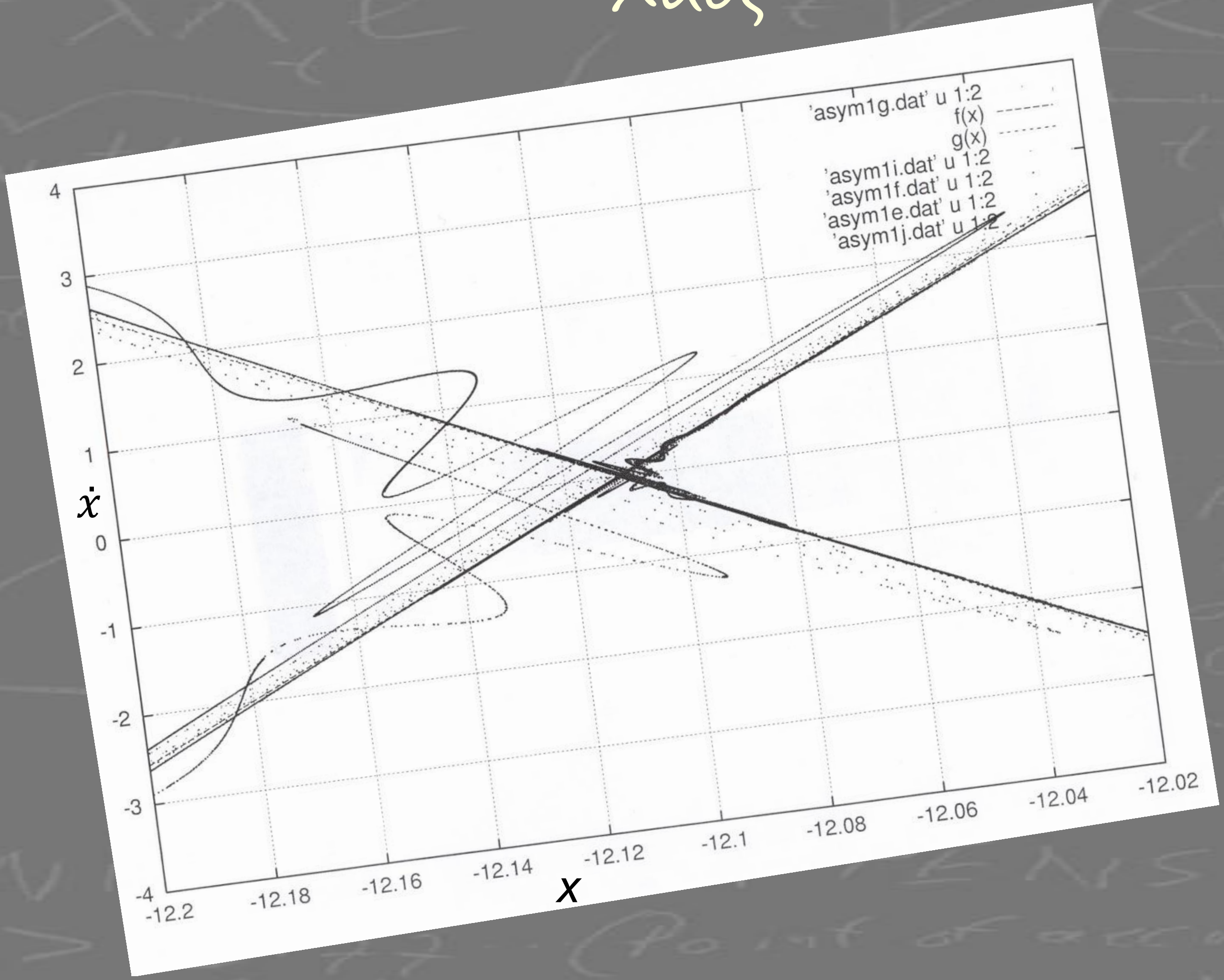
$|\alpha|>1$ UNSTABLE



Ελλειπτικά σημεία

Υπερβολικά σημεία

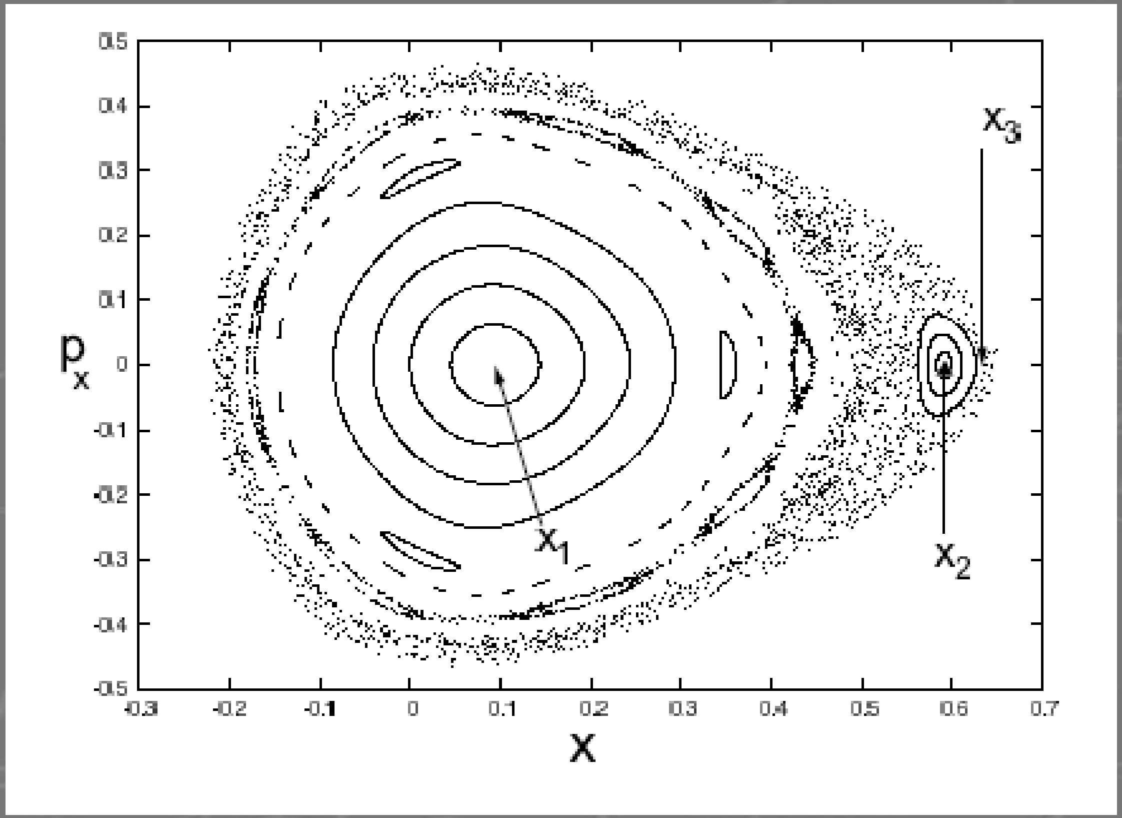
Χάος



'asym1g.dat' u 1:2
f(x)
g(x)
'asym1i.dat' u 1:2
'asym1f.dat' u 1:2
'asym1e.dat' u 1:2
'asym1j.dat' u 1:2
$\dot{x}$
x

The role of periodic orbits

Order +Chaos (2D case)



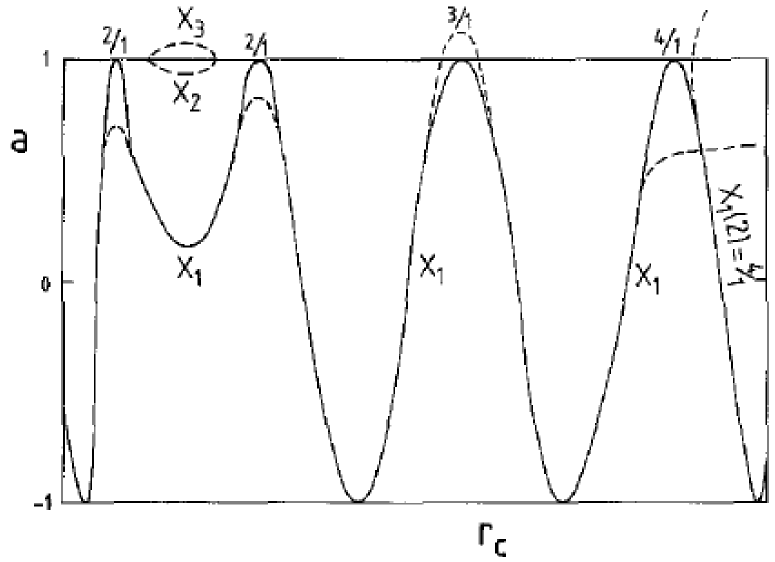
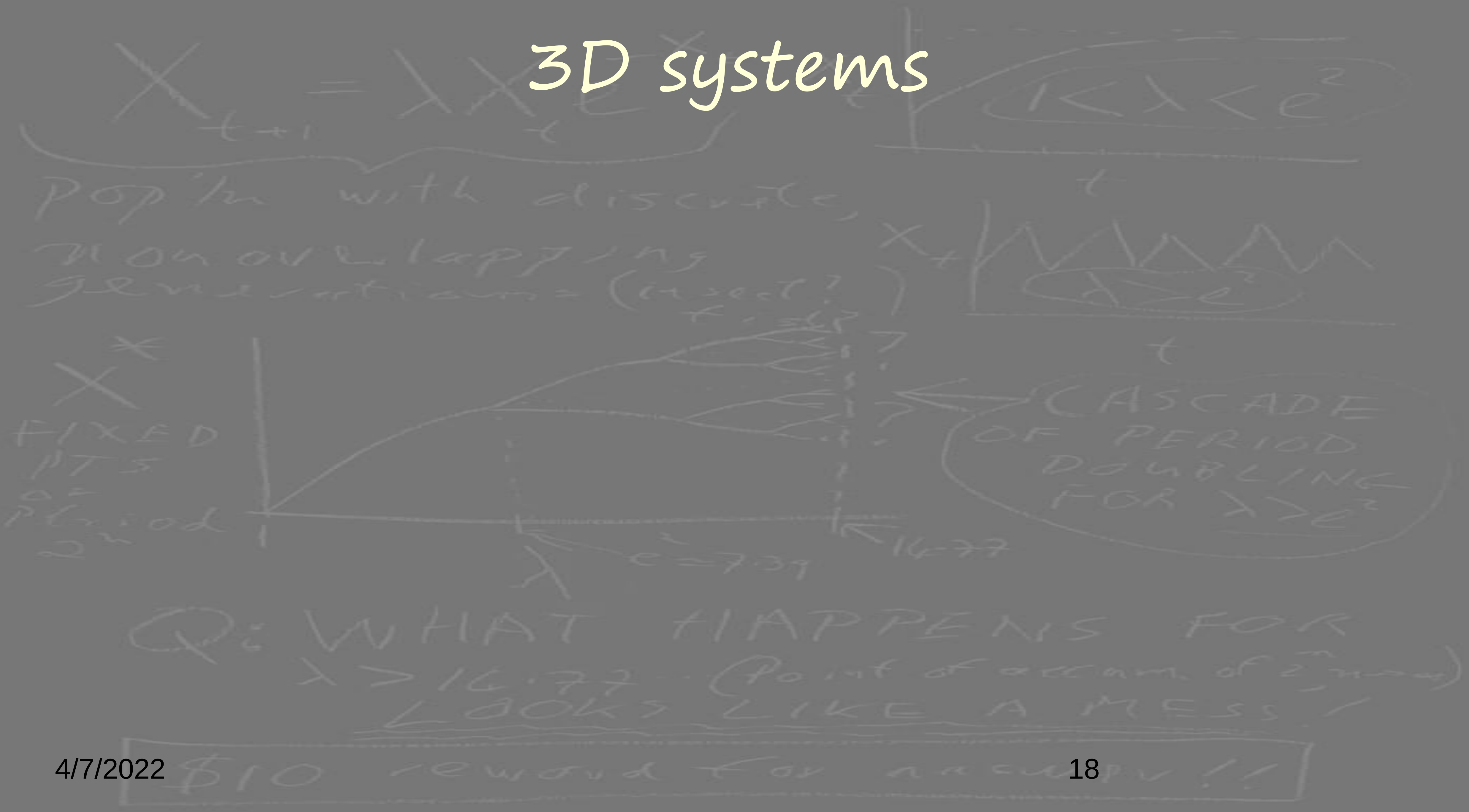


Fig. 5. Stability curves for a model with a double inner Lindblad resonance (schematically). (——) $A=0$ (axisymmetric case), (----) $A\neq 0$

3D systems


pop'ln with discrete, non overlapping generations (insect?)
FIXED PTS OF period 2^n
CASCADE OF PERIOD DOUBLING FOR $\lambda > e^2$
$e^2 = 7.39$
14.77
Q: WHAT HAPPENS FOR
LOOKS LIKE A MESS
$10 reward for answer!!

Φ=Miyamoto disk + Plummmer sphere + 3D Ferrers bar

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \Phi(x, y, z) - \Omega_b(xp_y - yp_x),$$

with

$$\Phi(x, y, z)_{eff} = \Phi(x, y, z) - \Omega_b(xp_y - yp_x)$$

$$\dot{x} = p_x + \Omega_b y, \qquad \dot{y} = p_y - \Omega_b x, \qquad \dot{z} = p_z$$

$$\dot{p}_x = -\frac{\partial \Phi}{\partial x} + \Omega_b p_y, \quad \dot{p}_y = -\frac{\partial \Phi}{\partial y} - \Omega_b p_x, \quad \dot{p}_z = -\frac{\partial \Phi}{\partial z}$$

$$\Phi(x, y, z) = \Phi_D + \Phi_S + \Phi_B$$

4D space of section, i.c. (x, p_x, z, p_z) in the plane $y=0$ with $p_y>0$

Application in a 3D rotating galactic potential

$$\Phi_D = -\frac{GM_D}{\sqrt{x^2 + y^2 + (A + \sqrt{B^2 + z^2})^2}},$$

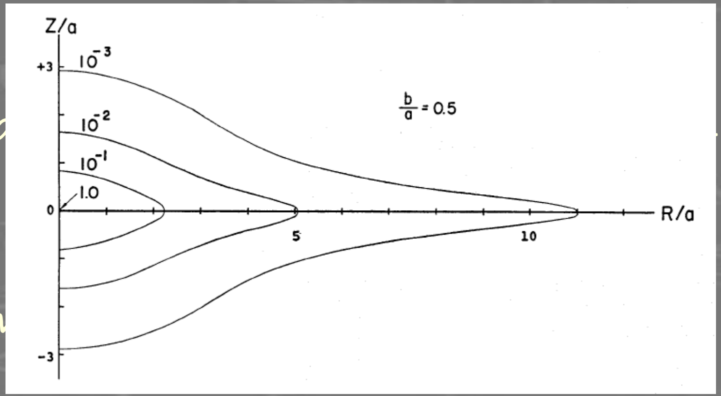
$$\Phi_S = -\frac{GM_S}{\sqrt{x^2 + y^2 + z^2 + \epsilon_S^2}},$$

$$\rho = \begin{cases} \frac{105 M_B}{32\pi abc}(1 - m^2)^2 & \text{for} \quad m \leq 1 \\ 0 & \text{for} \quad m > 1 \end{cases},$$

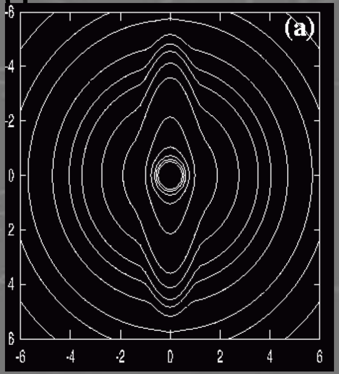
where

$$m^2 = \frac{y^2}{a^2} + \frac{x^2}{b^2} + \frac{z^2}{c^2}, \quad a > b > c,$$

- Miya
- Plun



- Ferrers bar, a:b:c = 6:1.5:0.6



Linear Stability

The relation of the final deviations of this neighboring orbit from the periodic one, with the initially introduced deviations can be written in vector form as: $\vec{\xi} = M\,\vec{\xi}_0$. Here $\vec{\xi}$ is the final deviation, $\vec{\xi}_0$ is the initial deviation and M is a 4×4 matrix, called the monodromy matrix. It can be shown that the characteristic equation is written in the form $\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \alpha\lambda + 1 = 0$. Its solutions $(\lambda_i, i = 1, 2, 3, 4)$ obey the relations $\lambda_1\,\lambda_2 = 1$ and $\lambda_3\,\lambda_4 = 1$ and for each pair we can write:

$$\lambda_i, 1/\lambda_i = \frac{1}{2}[-b_i \pm (b_i^2 - 4)^{\frac{1}{2}}],$$

where $b_i = 1/2\,(\alpha \pm \Delta^{1/2})$ and

stability indices

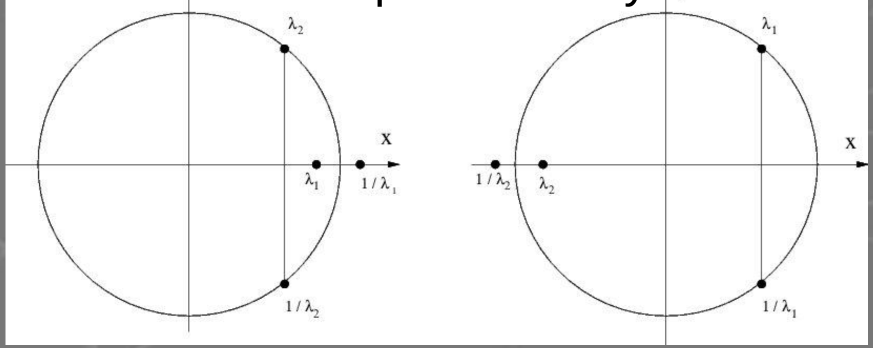
$$\Delta = \alpha^2 - 4(\beta - 2).$$

motion is stable when all the roots of (44) are complex conjugate lying on the unit circle, and this happens when the following three inequalities hold:

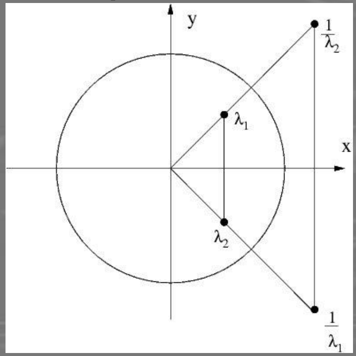
$$\Delta > 0, \qquad |b_1| < 2, \qquad |b_2| < 2. \tag{49}$$

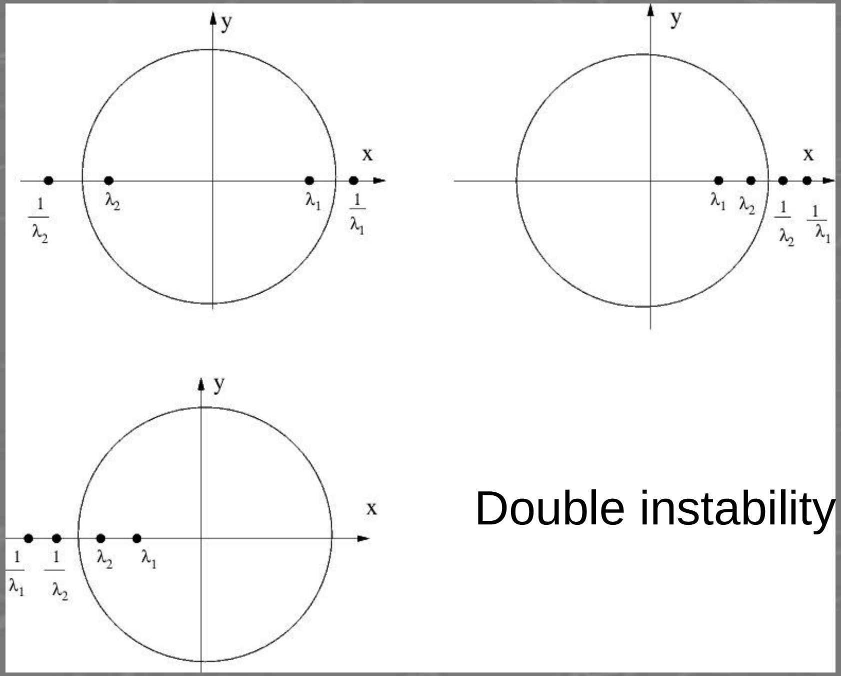
In all other cases the motion is unstable.

Simple instability



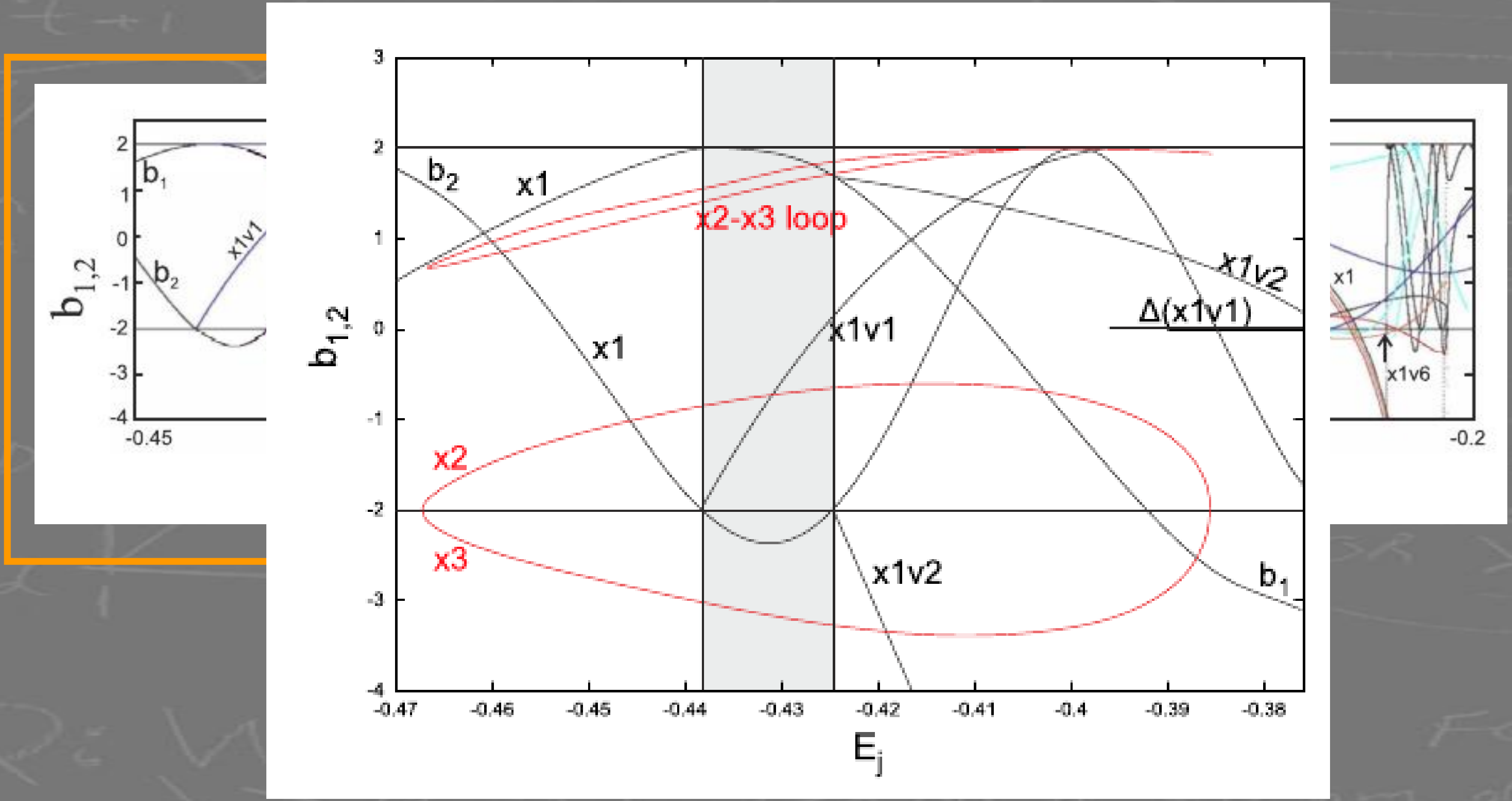
Complex instability





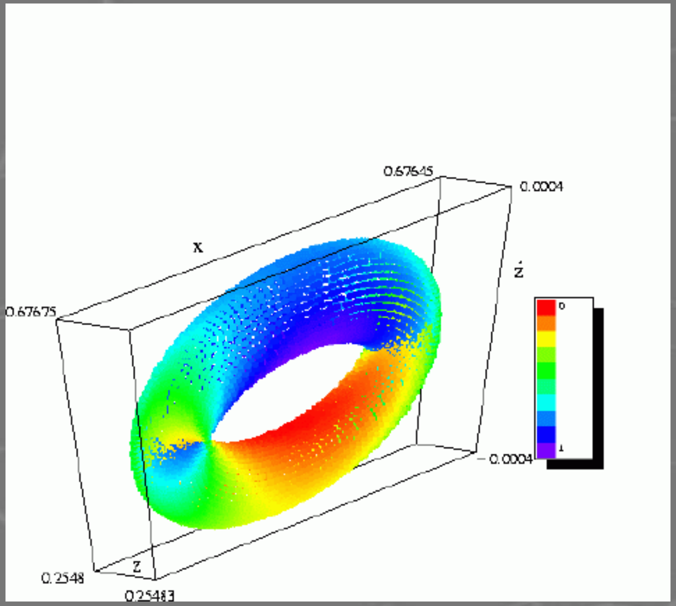
Double instability

Complex instability and the x1v1 family

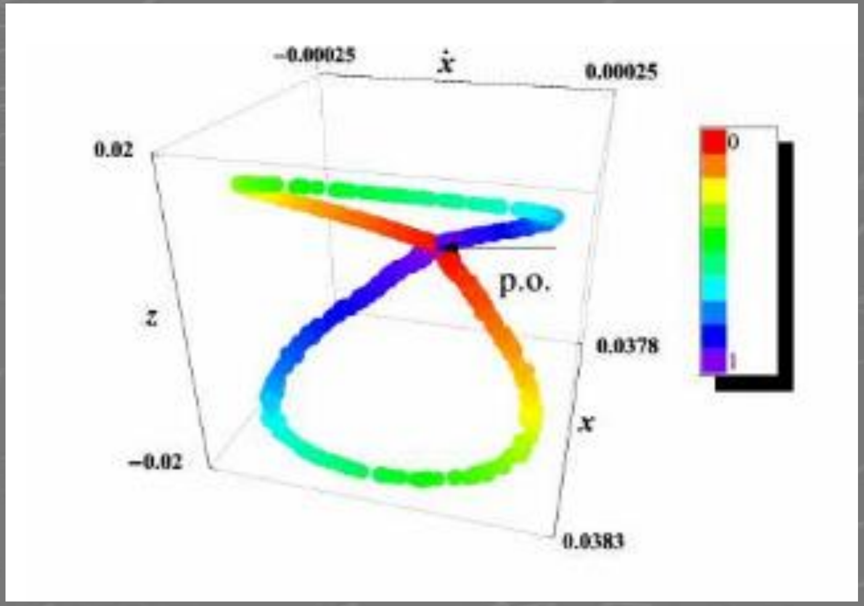


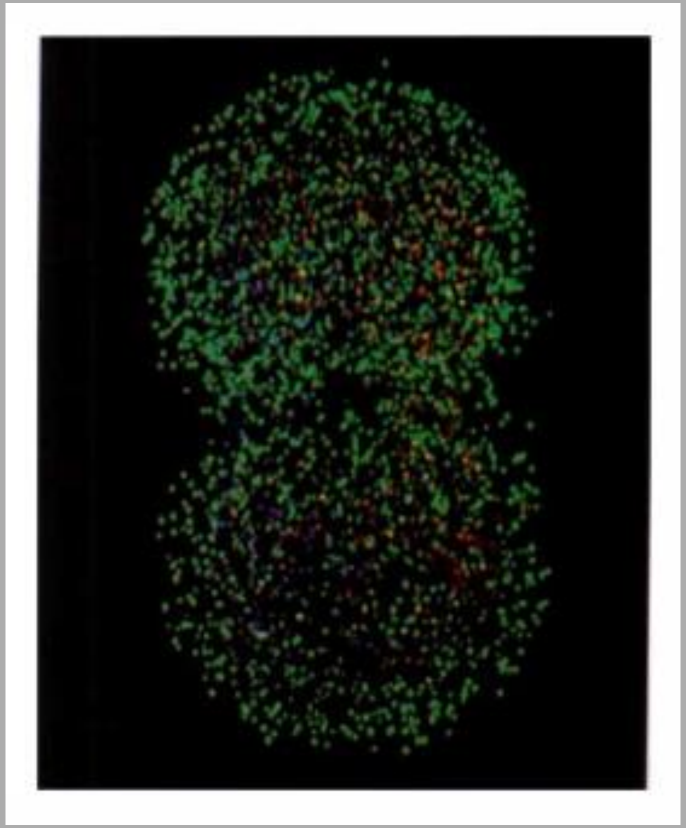
The structure of phase space in 3D systems visualization as in Patsis & Zachilas 1994 IJBC

Stability: Katsanikas & P. 2011



Simple Instability: P & Katsanikas 2014

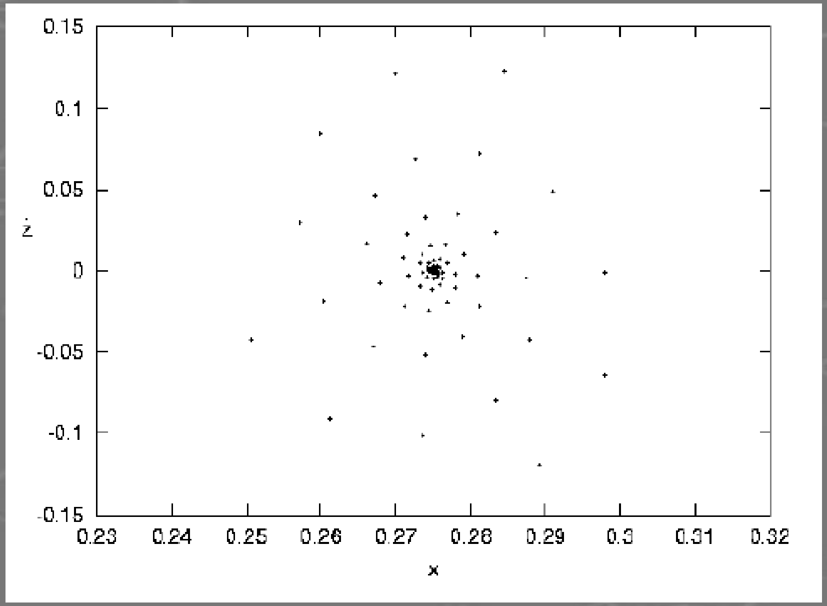




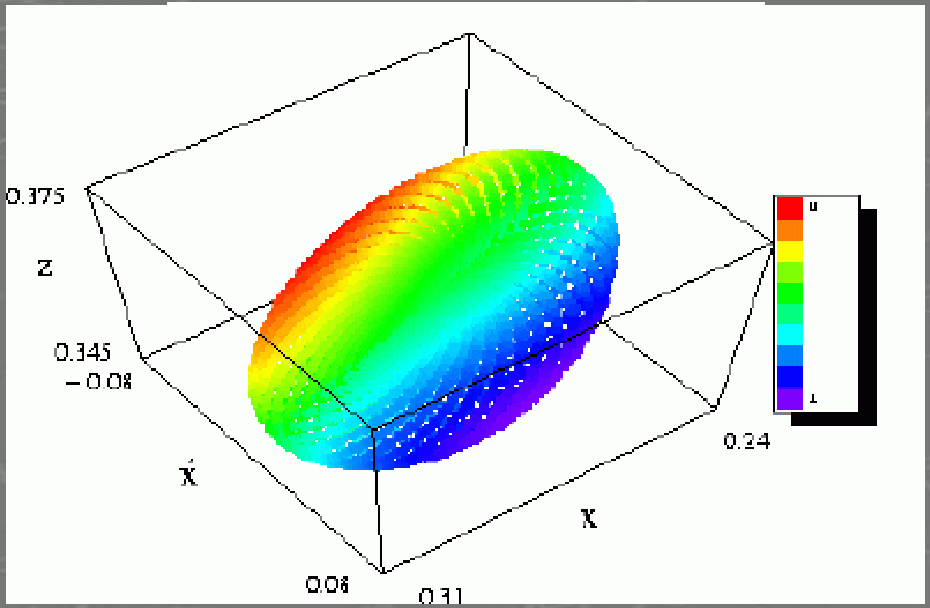
Double Instability
P. & Zachilas 1994
IJBC 4, 1399

What do we know about the neighborhood of complex unstable periodic orbits?

Contontopoulos, Farantos, Papadaki, Polymilis 1994

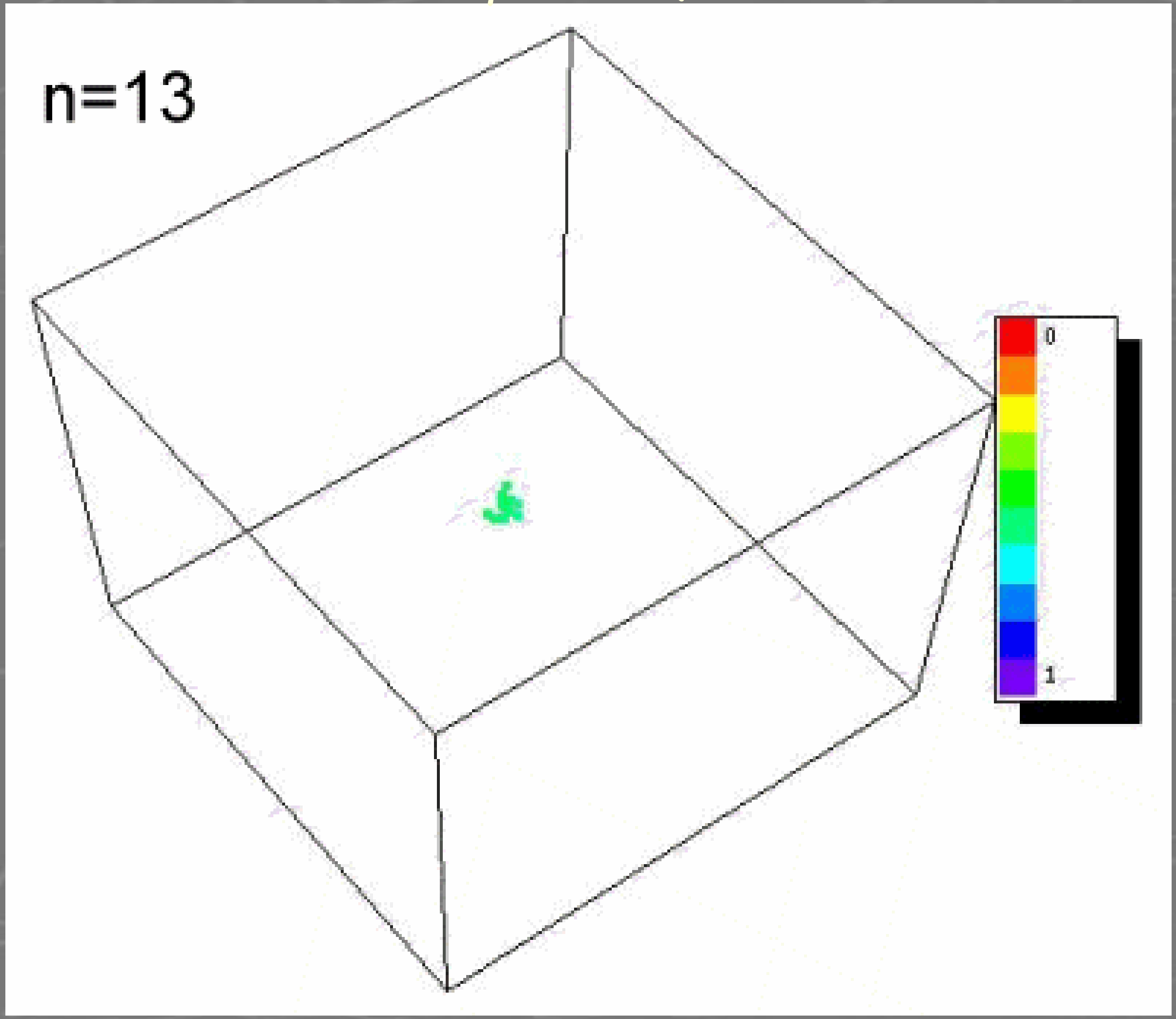


Katsanikas, Patsis, Contopoulos

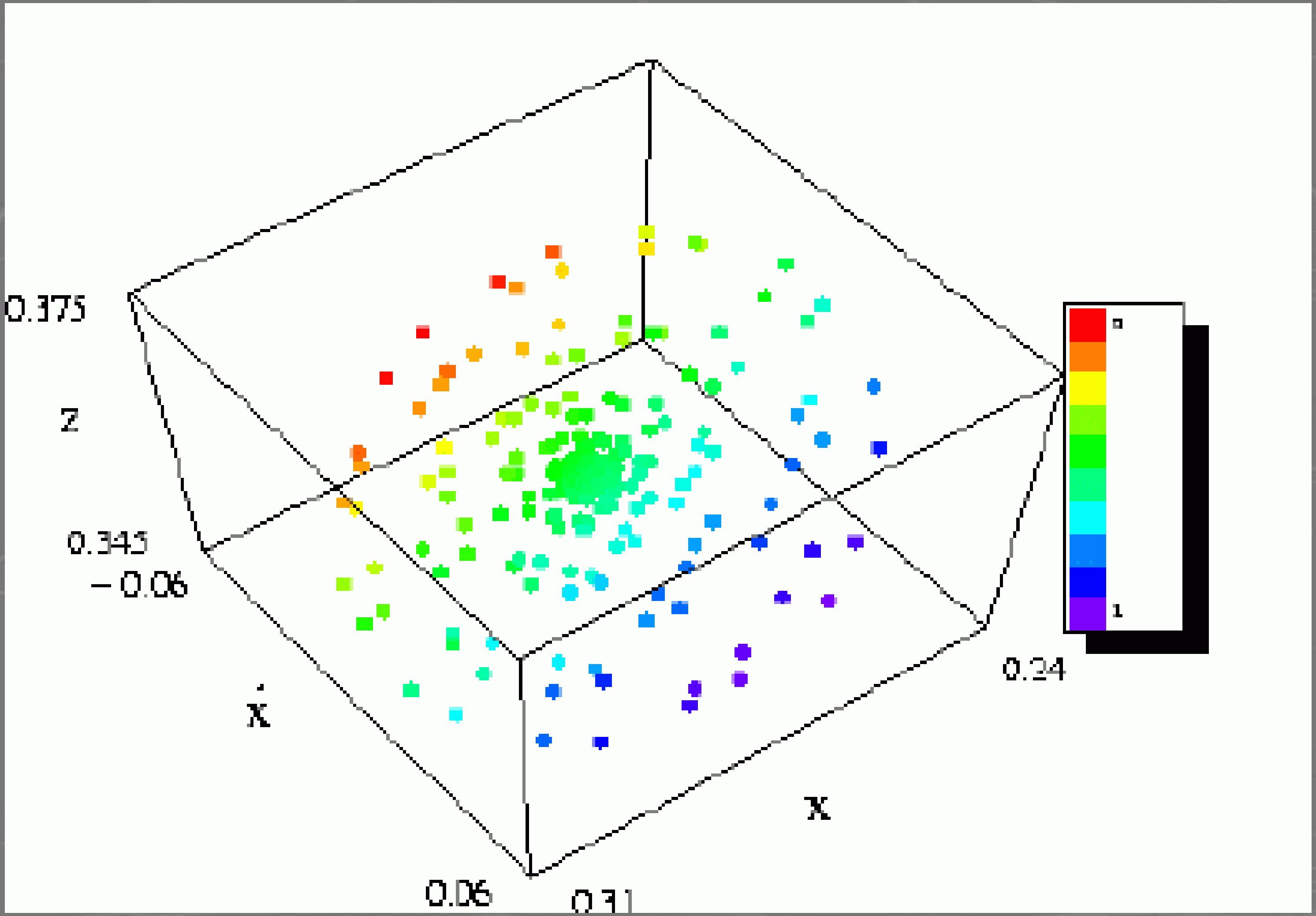


"confined torus" Pfenniger 1984

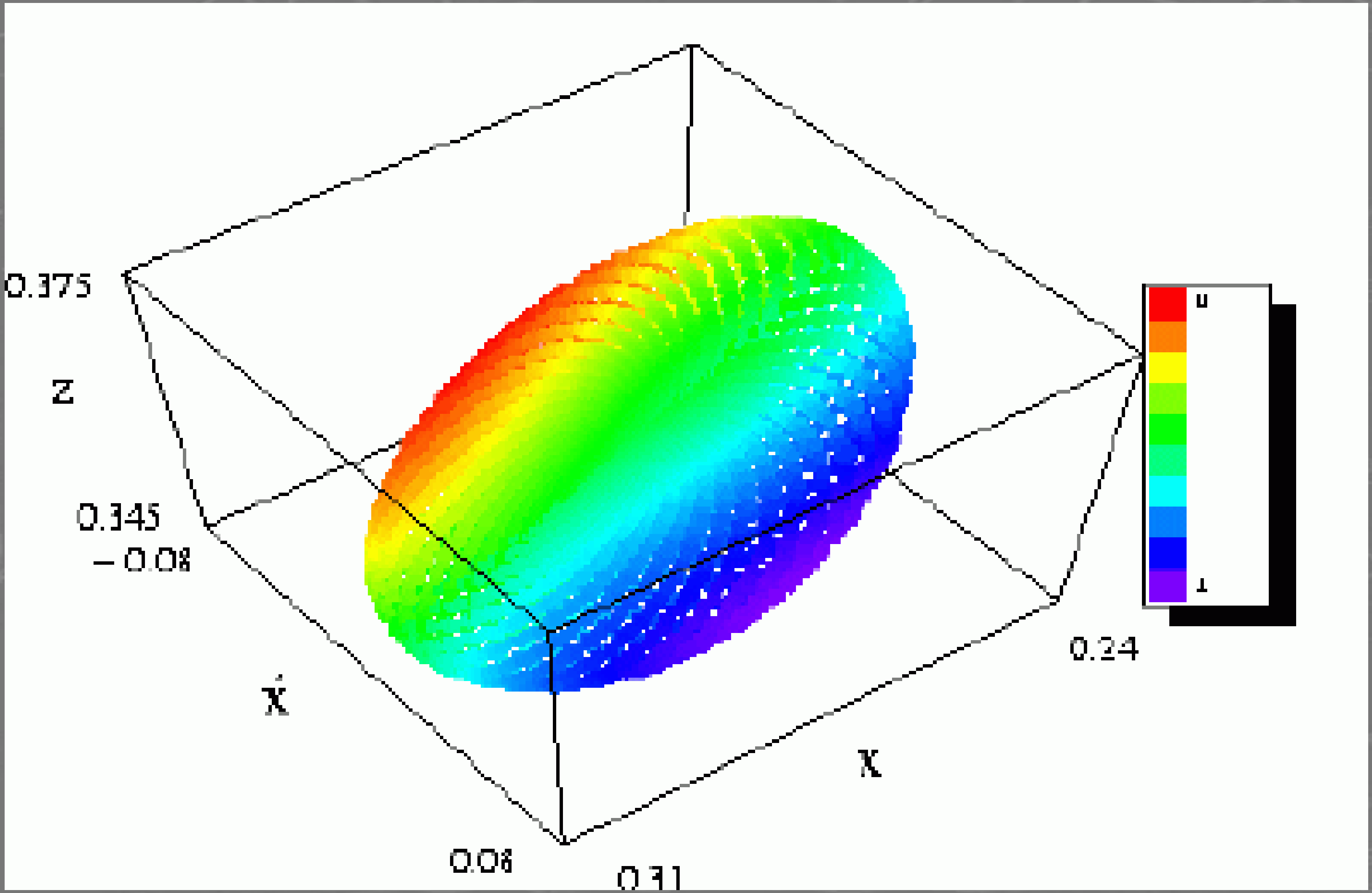
Complex instability –
Katsanikas, Patsis, Contopoulos (2011, IJBC 21, 2321)



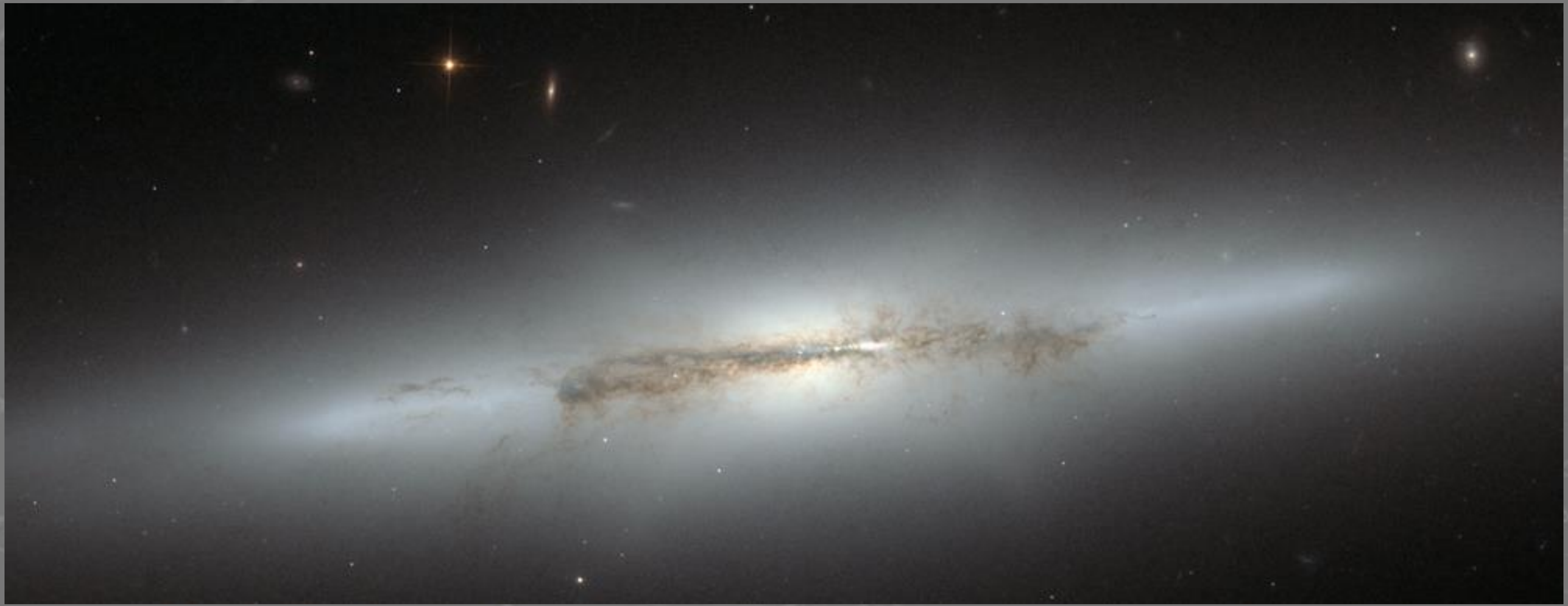
Complex instability



Complex instability – confined torus

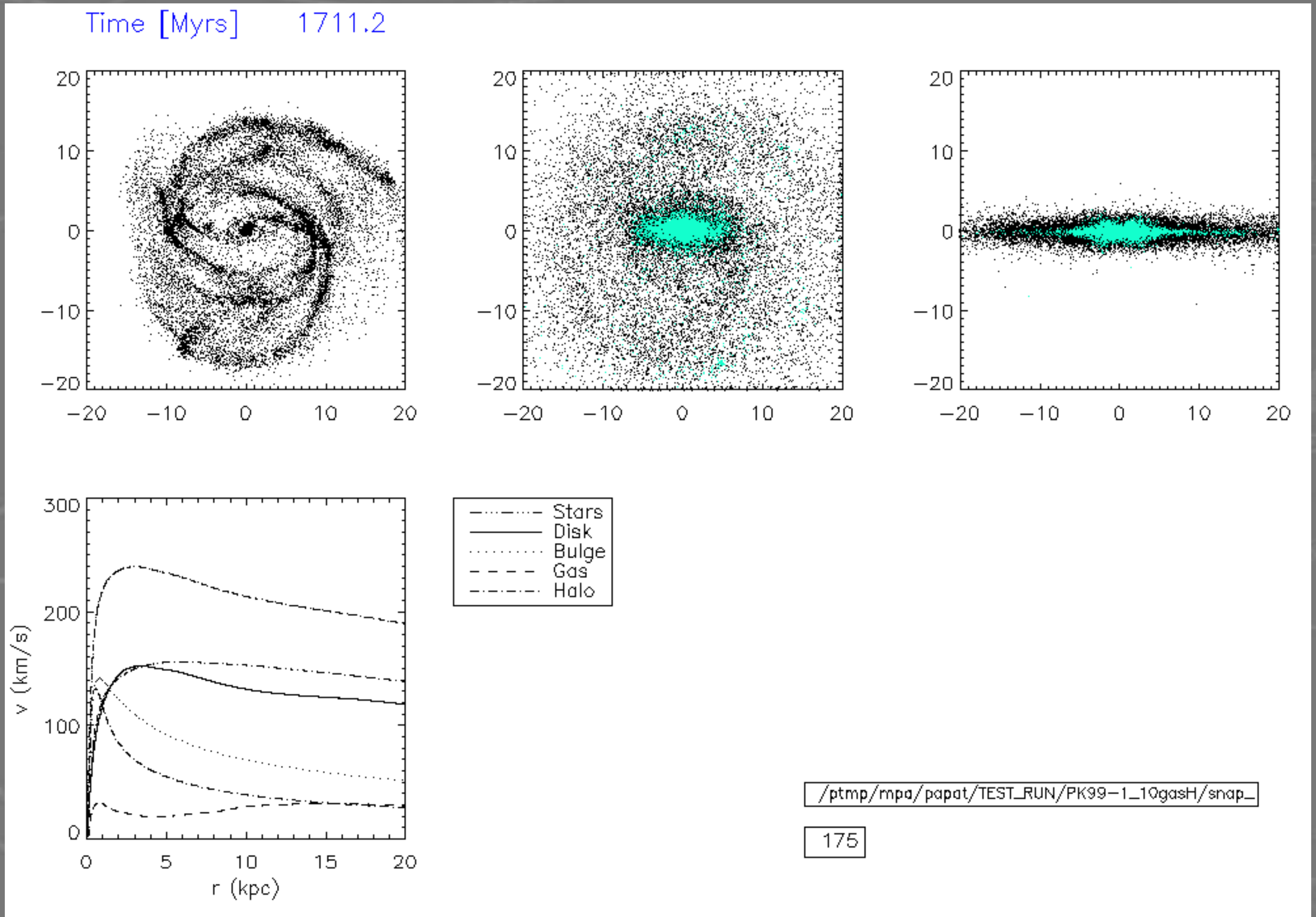


NGC 4710, $\alpha=12^{h}\ 49^{m}\ 38.9$, $\delta=+15^{\circ}\ 9'\ 56''$

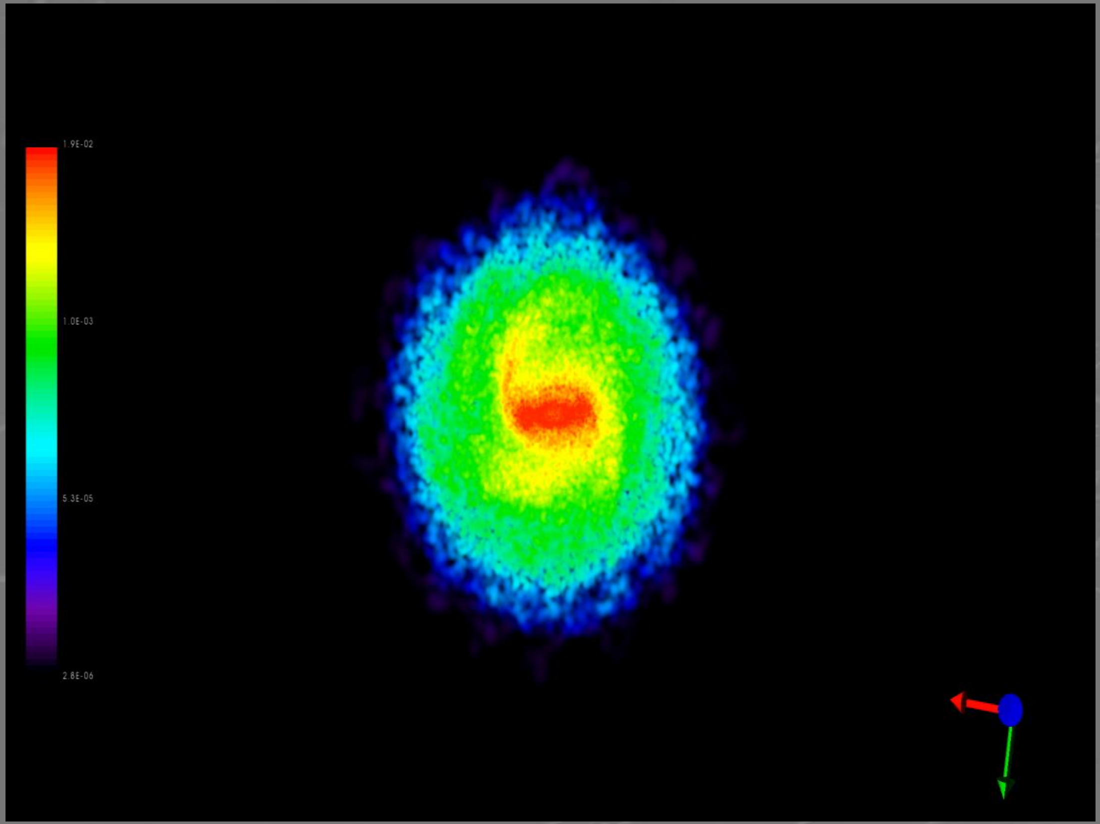


This natural-color photo was taken with the Hubble Space Telescope's Advanced Camera for Surveys on January 15, 2006

N-body peanuts II

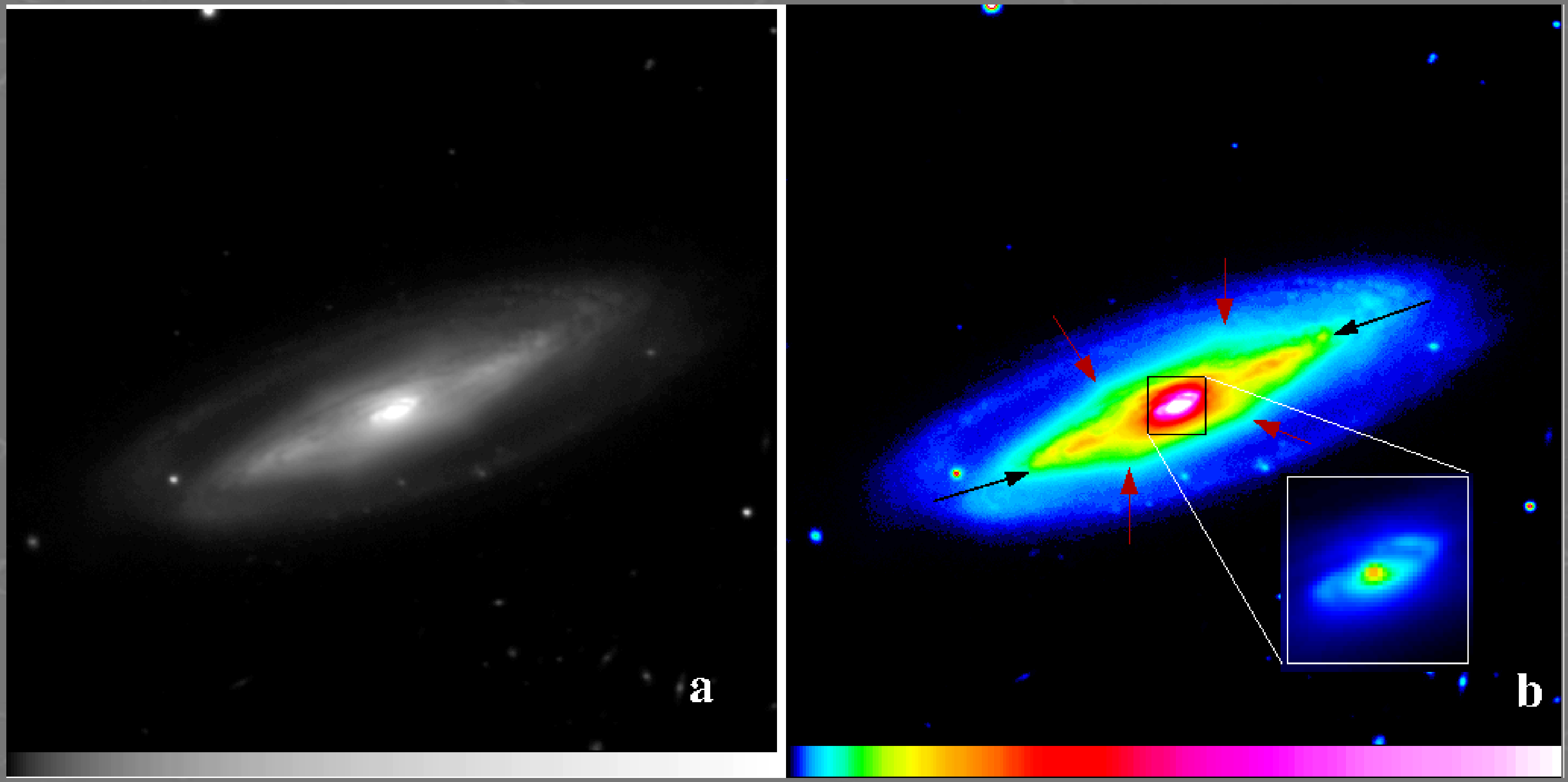


s/s from GADGET3 N-body simulation (Patsis & Naab 2022 – in preparation)



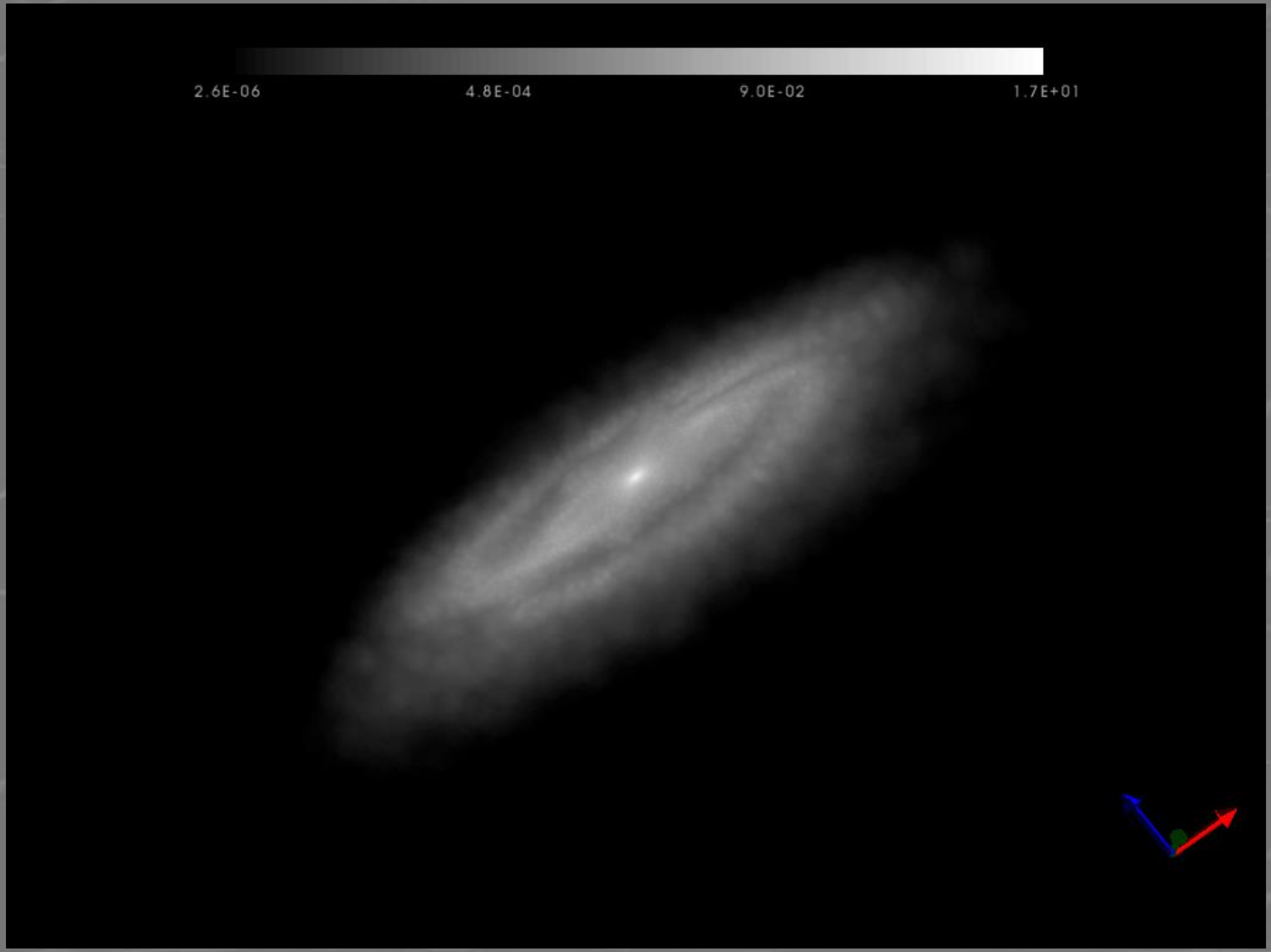
2 x 10^6 particles (DM, stars, gas, newborn stars)

NGC352 (Aristarchos telescope, Helmos, Greece)

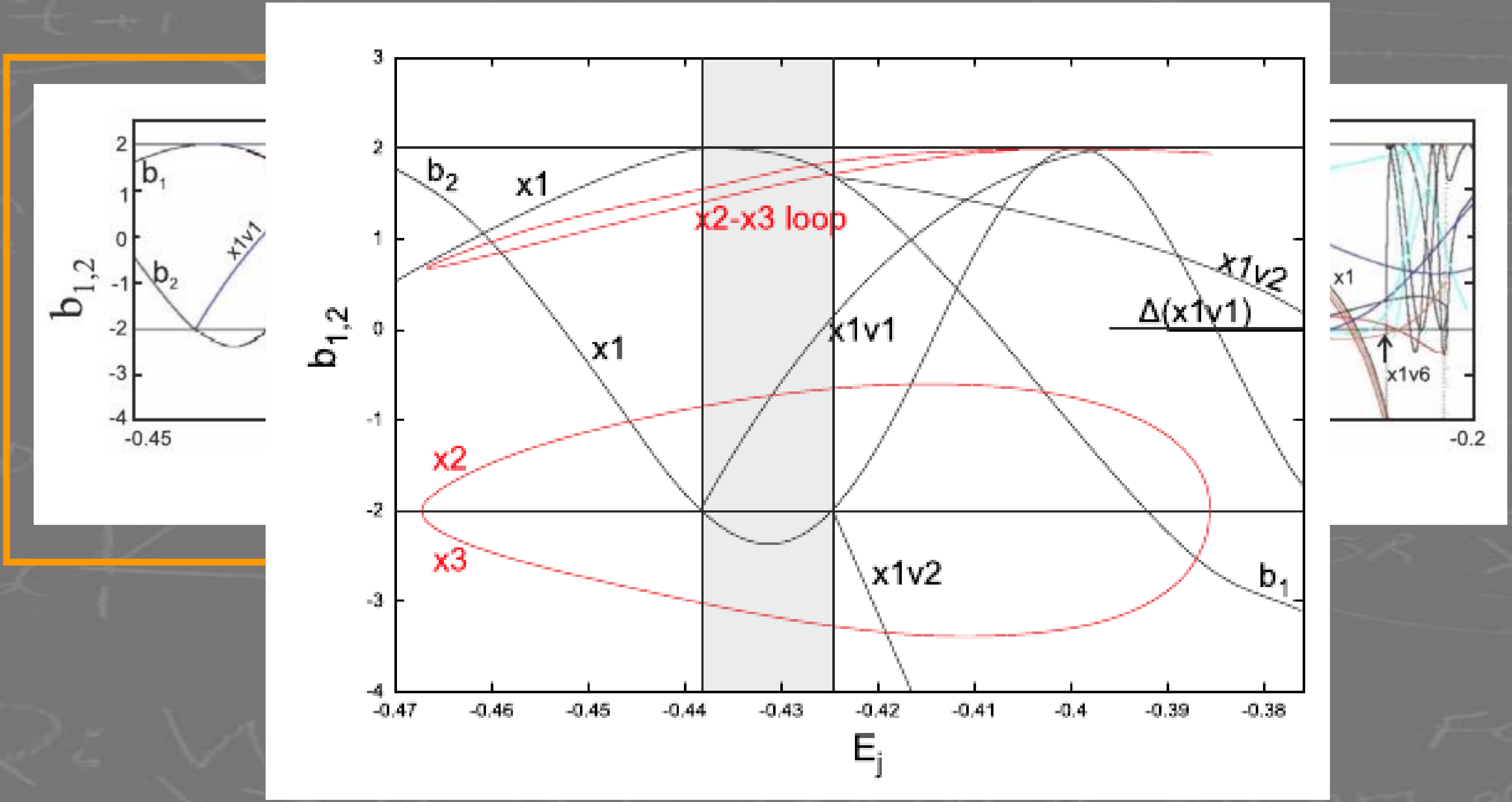


R filter. Patsis, Xilouris, Alikakos

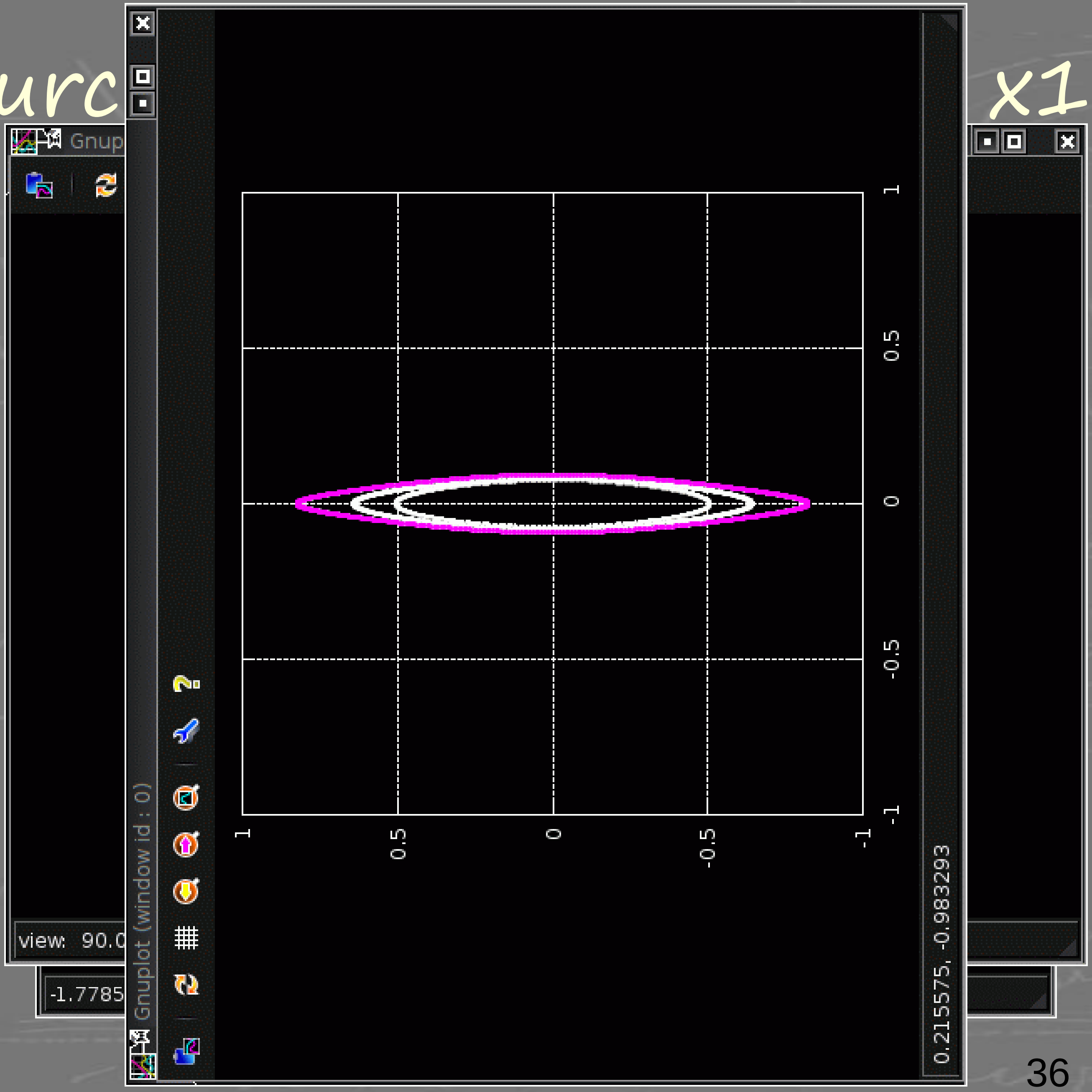
N-body s/s (Athanassoula 2017)



1. Where does the b/p start?

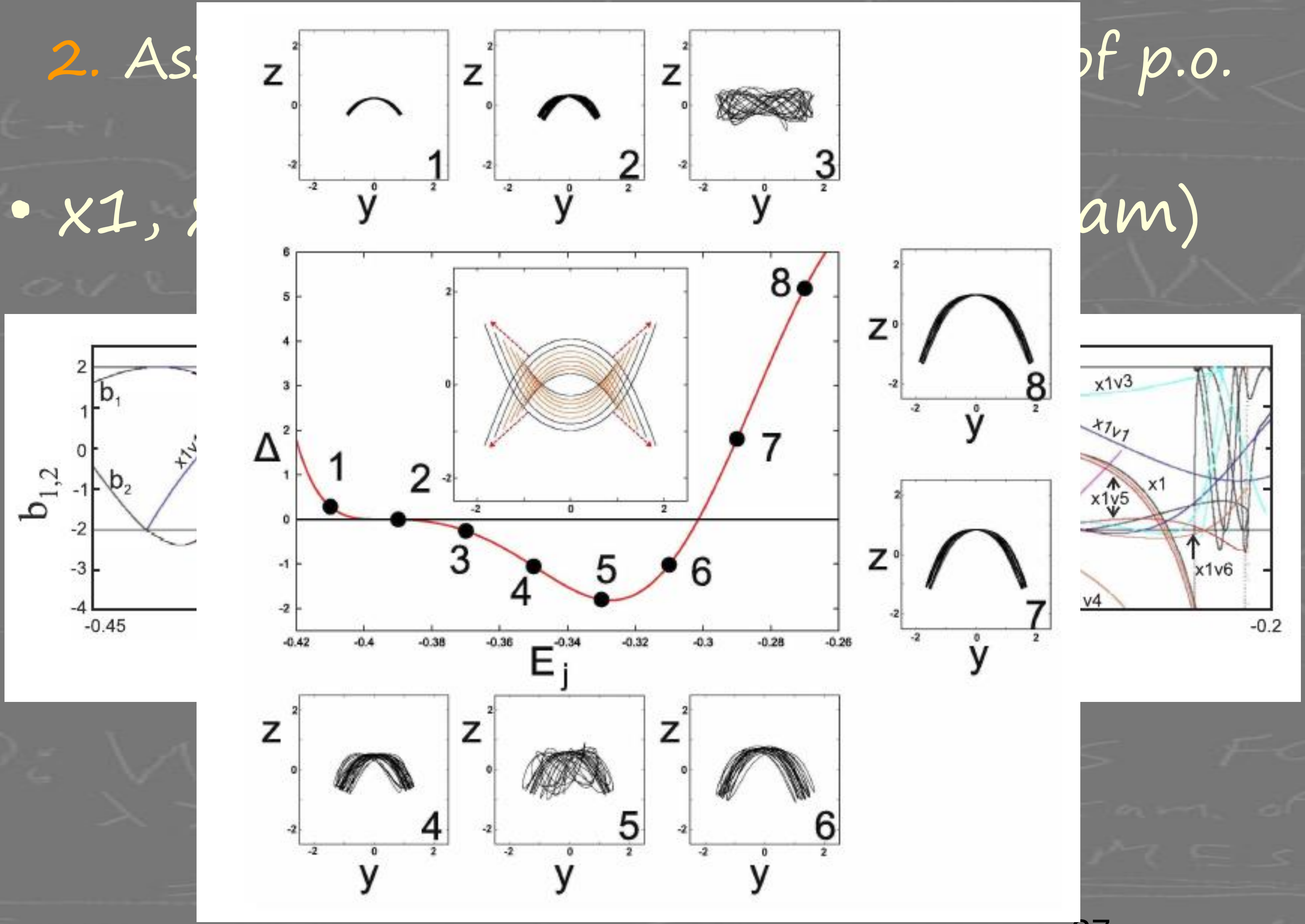


Bifurc … x1v1’

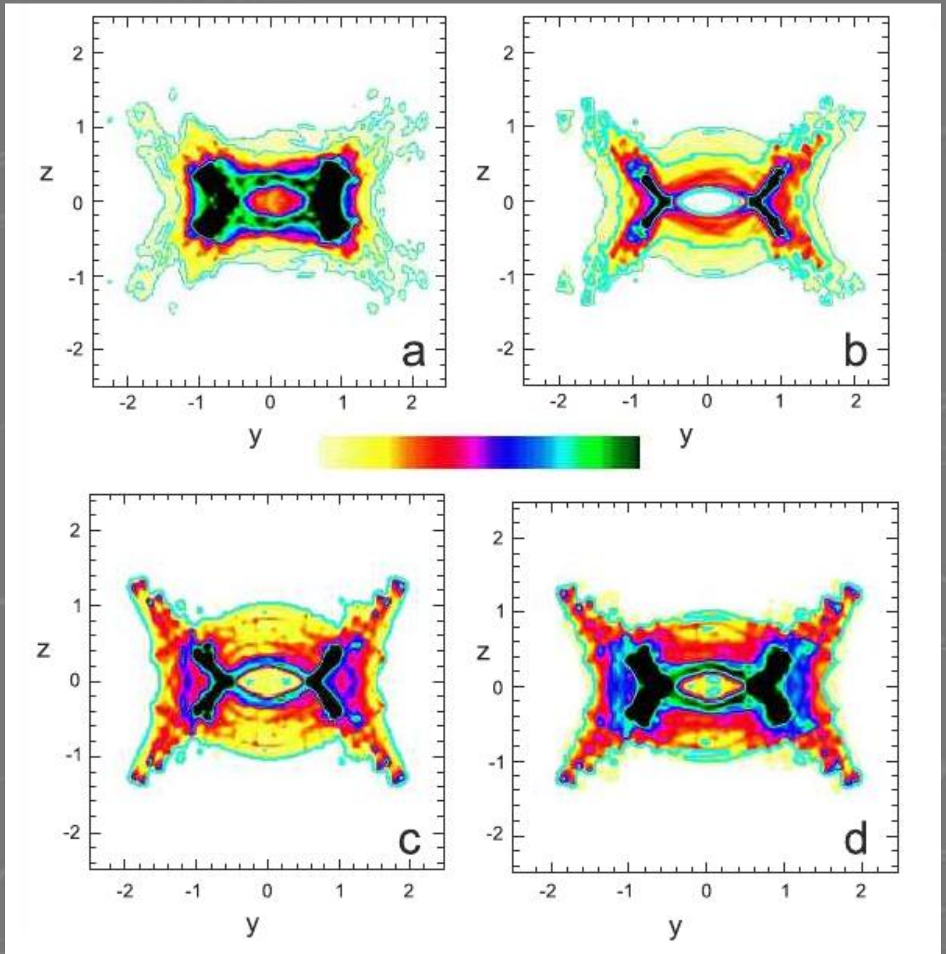


2. As... of p.o.

- x1, ... am)

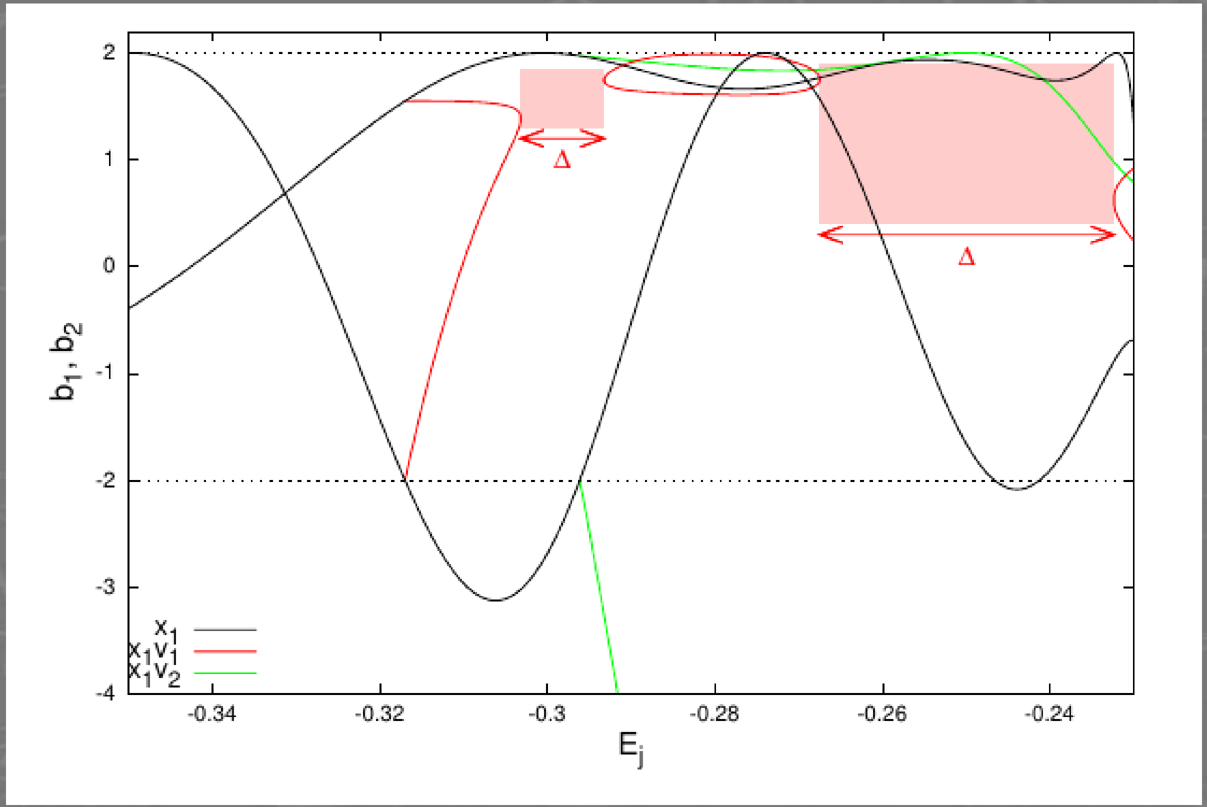


the "x1v1" scenario

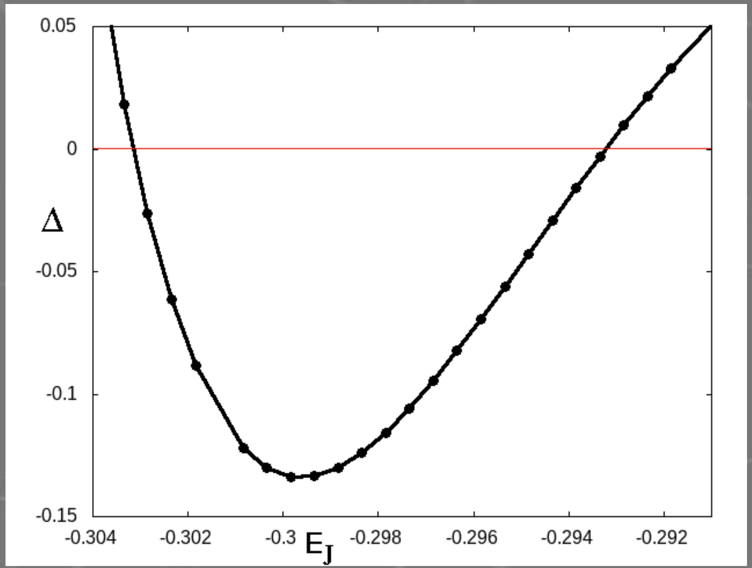


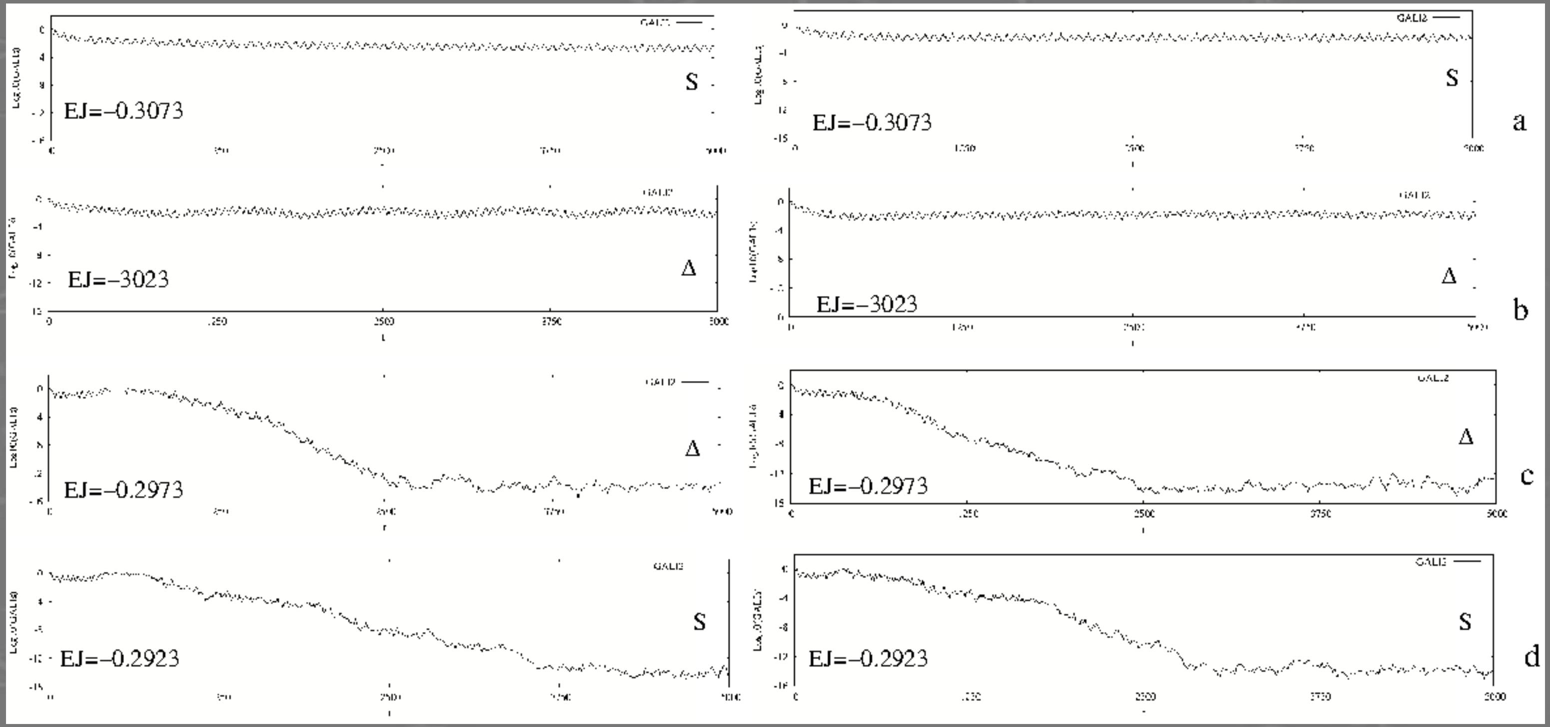
What can we build with orbits close to Δ p.o. ?
PhyD 42933050 (2022)

- + T. Manos, H. Skokos, L. Chaves-Velasquez, I.Puerari



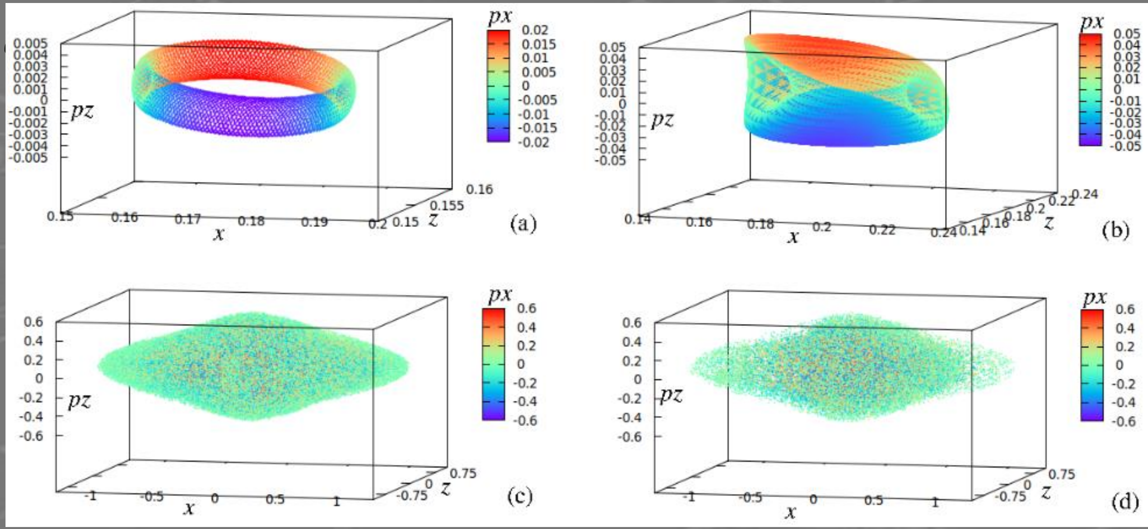
Orbits close to Δ p.o. in the first region.
“10%” perturbations

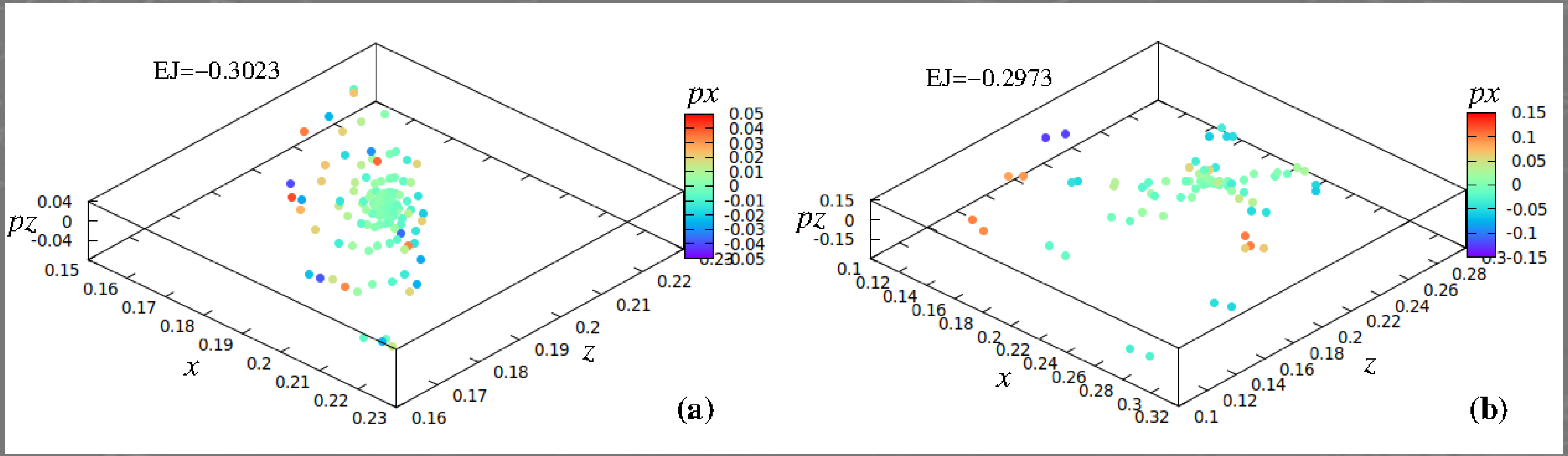


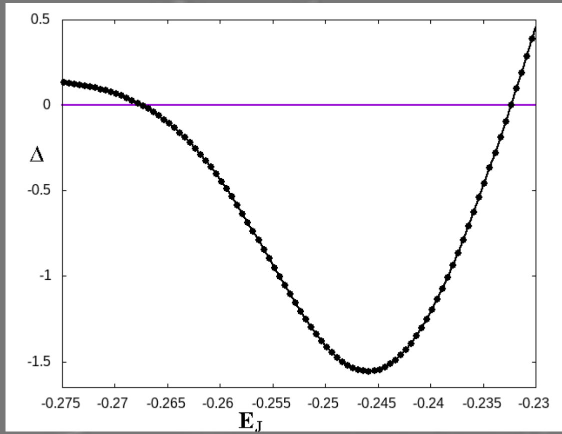


GALI2 index: Skokos, Bountis Antonopoulos 2007

Orbits close to Δ p.o. in the first region.

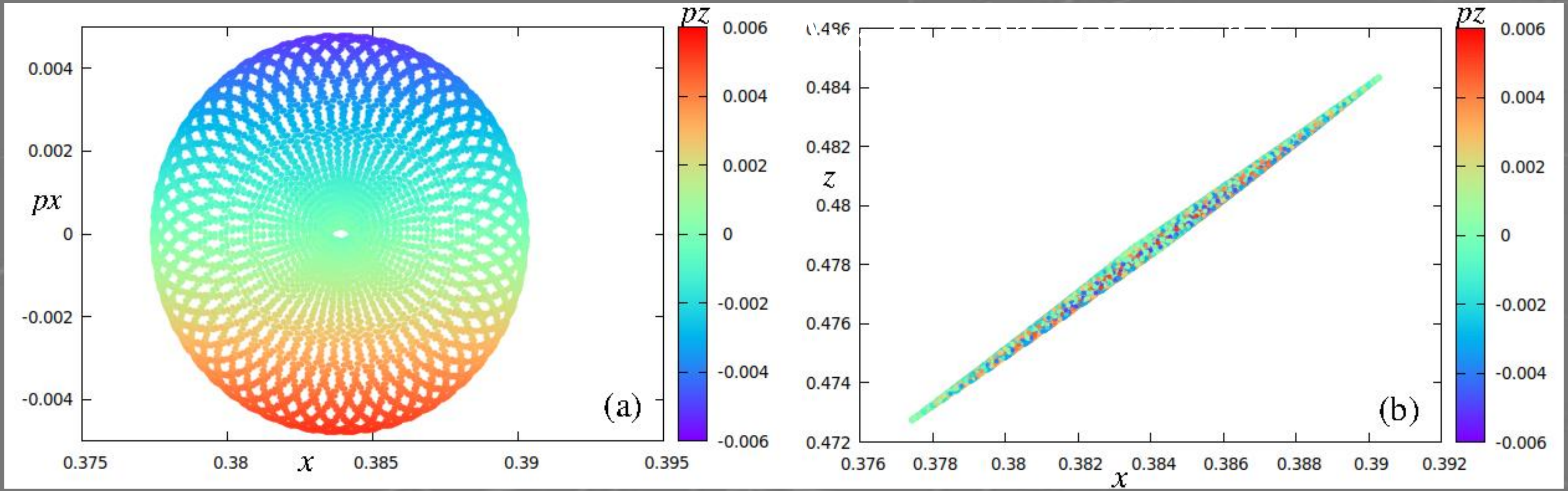






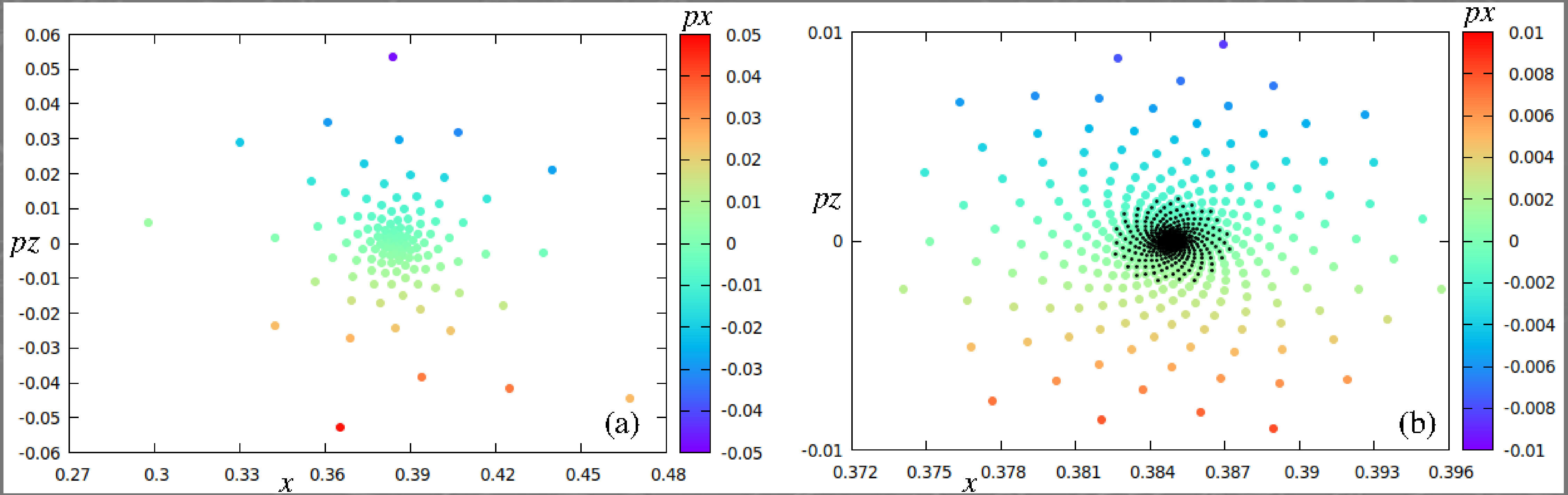
Orbits close to Δ p.o. in the SECOND region.
S: JUST before the S→Δ transition

EJ=-0.26753617



Orbits close to Δ p.o. in the SECOND region.
Δ: JUST AFTER the S→Δ transition

EJ=-0.26743617 Δx=0.001x0(x1v1) and Δx=10^{-8} x0(x1v1)

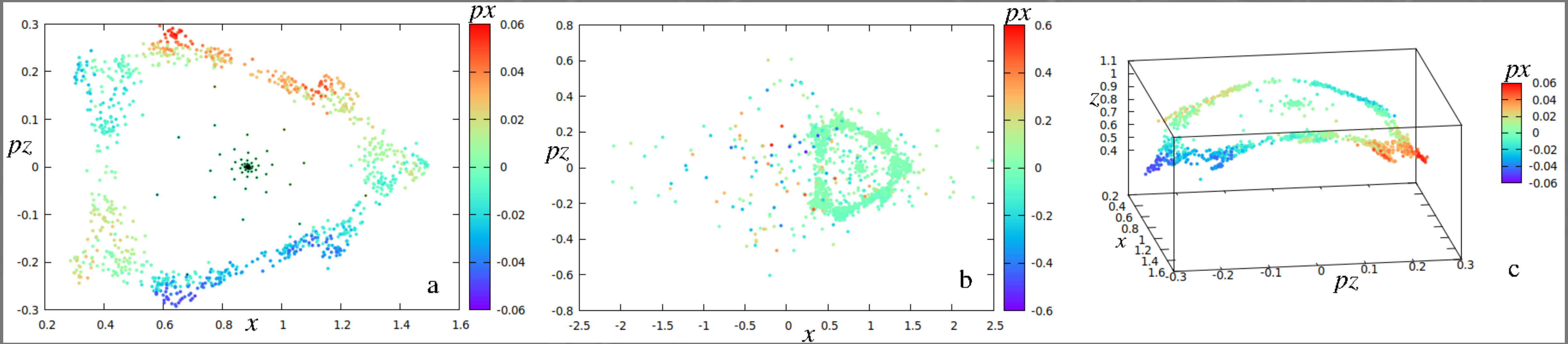


120
#

1800
(1600) #

Orbits close to Δ p.o. in the SECOND region. Δ: CLOSE to the Δ→S transition. The role of the ENVIRONMENT

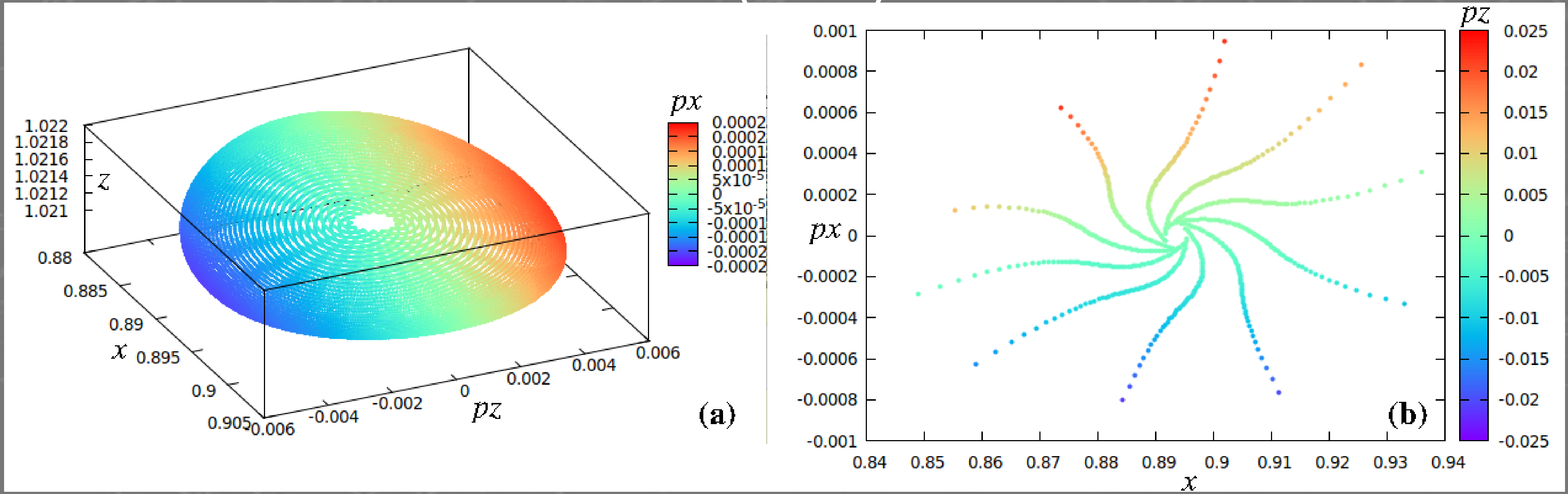
EJ=-0.23283617 Δx=10^{-8}x0(x1v1)



The same situation appears for smaller EJ (without spirals)

Orbits close to Δ p.o. in the SECOND region.
Δ: JUST AFTER the Δ→S transition.

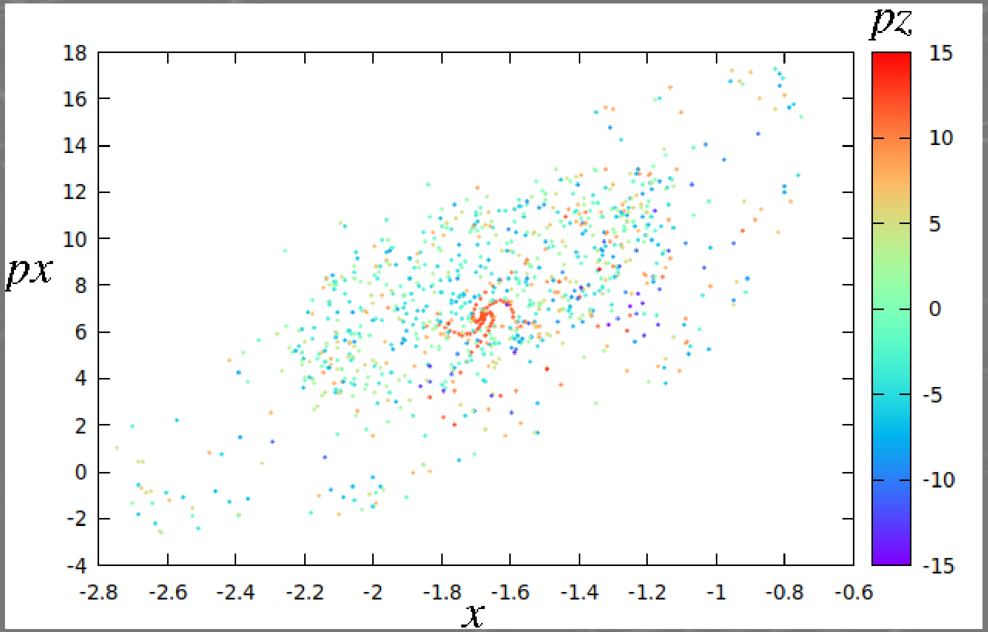
EJ=-0.23234 Δx=0.0015x0(x1v1) and



600# and then chaos

PERLAS potential. Similar behavior

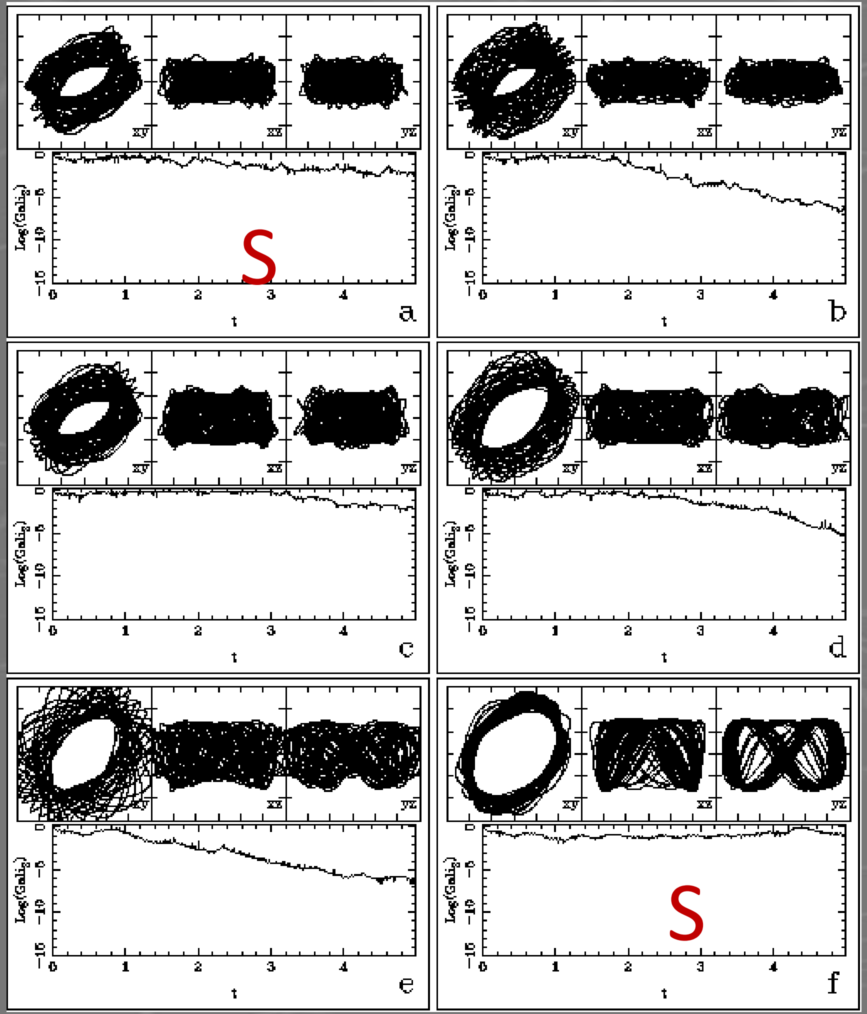
- The small Δ interval can be ignored
- In the large interval (e.g. EJ=-1208.228, Δx=0.001x0)…

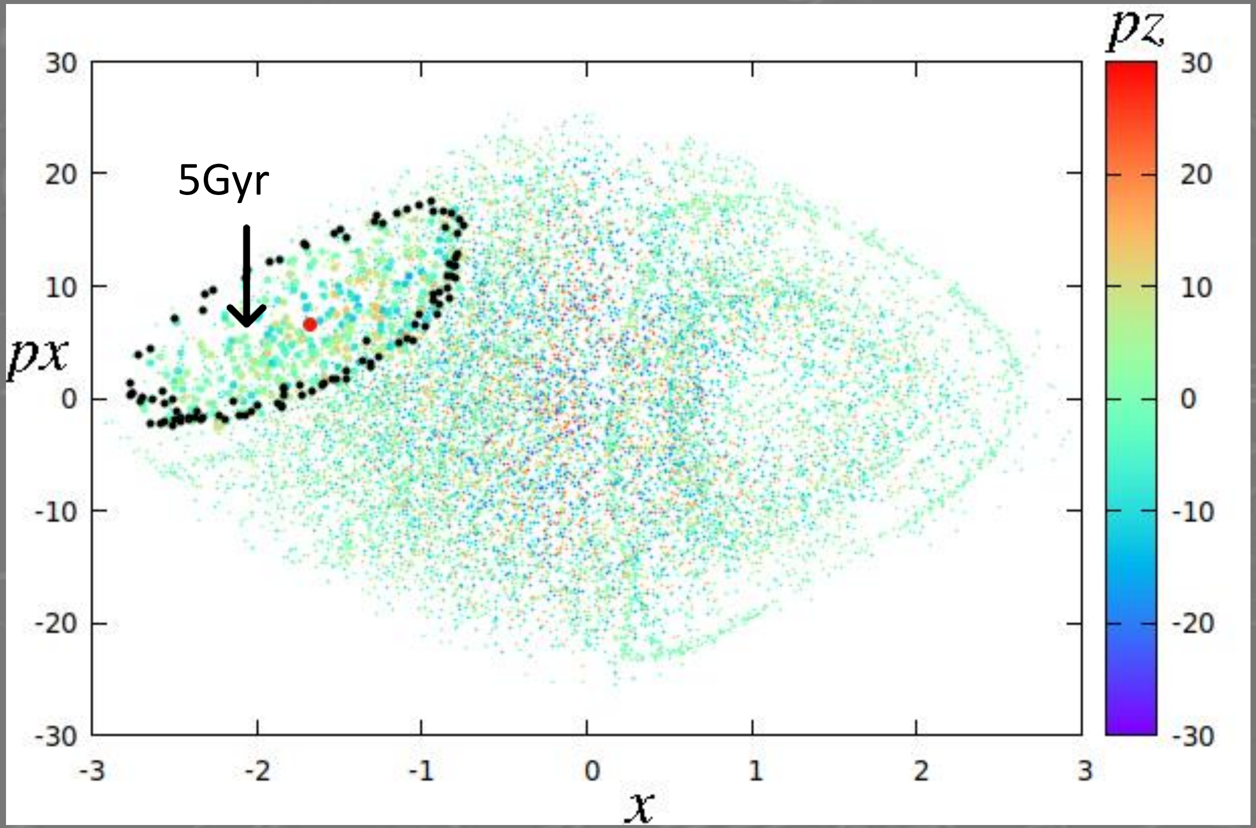


67# in the middle
Large dynamical time scales

PERLAS potential. Similar behavior

Perturbed 10% in x





Clockwise rotation (x0 i.c. with x<0)

CONCLUSIONS

- In a S→Δ→S transition there is a continuity in the evolution of the phase space in the immediate neighborhood of the periodic orbits:

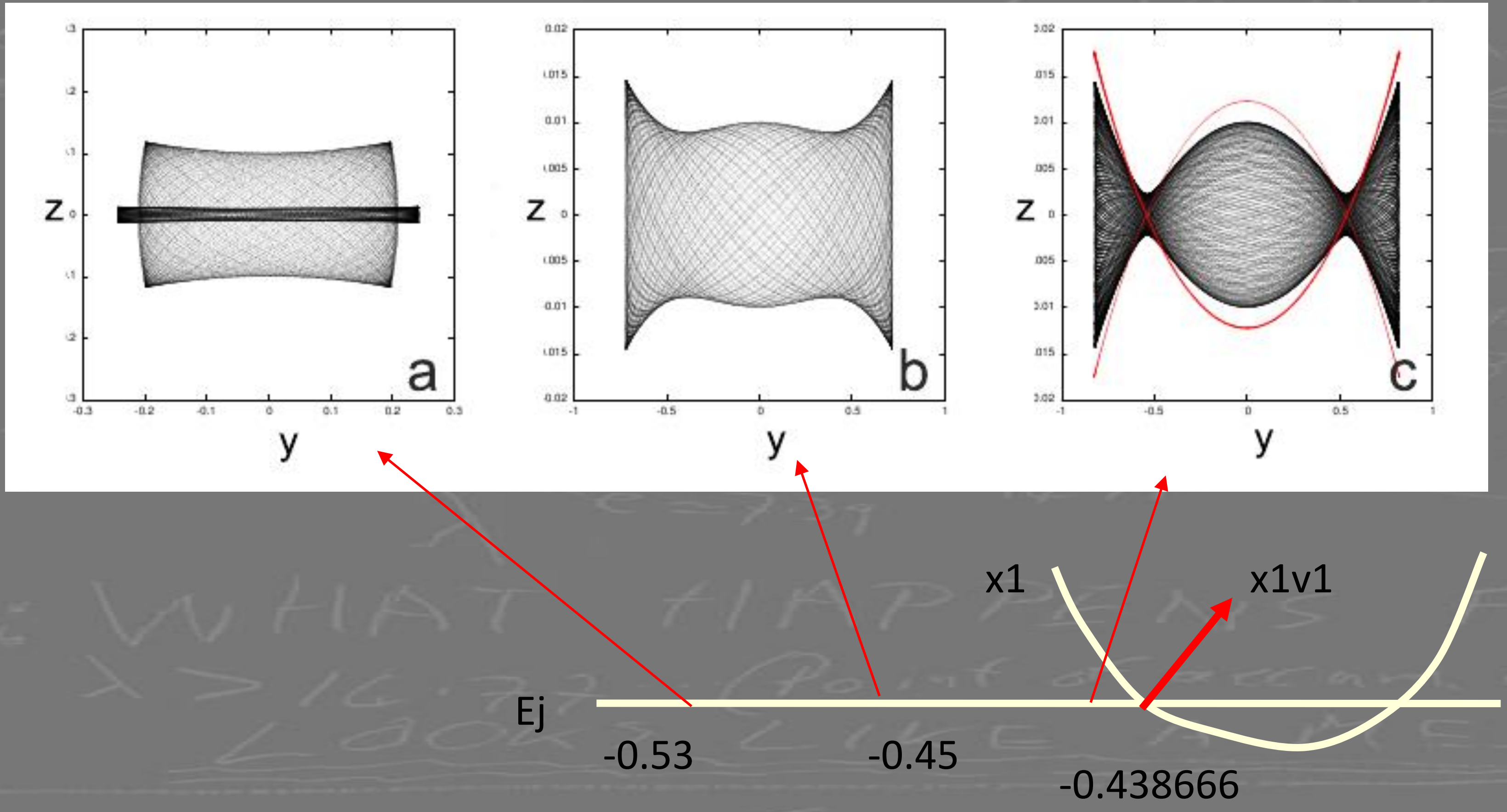
(S) shrinking of tori → (S) disky tori → (Δ) confined tori w. spirals →

(Δ) clouds w. spirals → (S) reverse evolution

- (this is the second example we are aware of, where the evolution of the phase space structure foretells an impending stability transition of a specific kind)

Early q-p x1 orbits (P&K2014)

x1v1



CONCLUSIONS

- Δ does not define the phase space structure
- The role of the close environment plays a significant role (even for trapping in sticky zones)
- Especially in PERLAS, a Δ region of x1v1 may rather increase locally the dispersion of velocities (weaken the spiral?) than destroy the pattern

MNRAS 509, 1995 (2022)

Orbits in time dependent potentials (+Manos, Skokos)

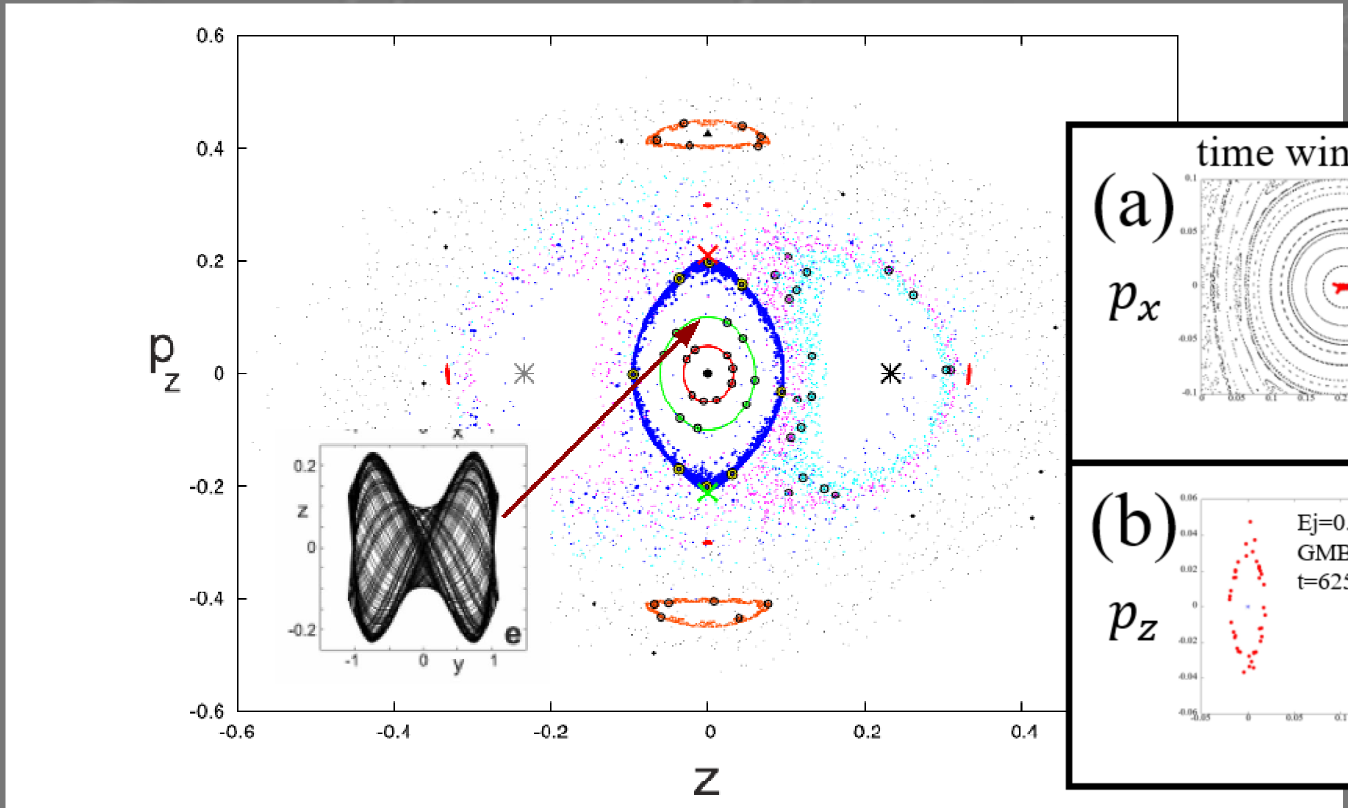
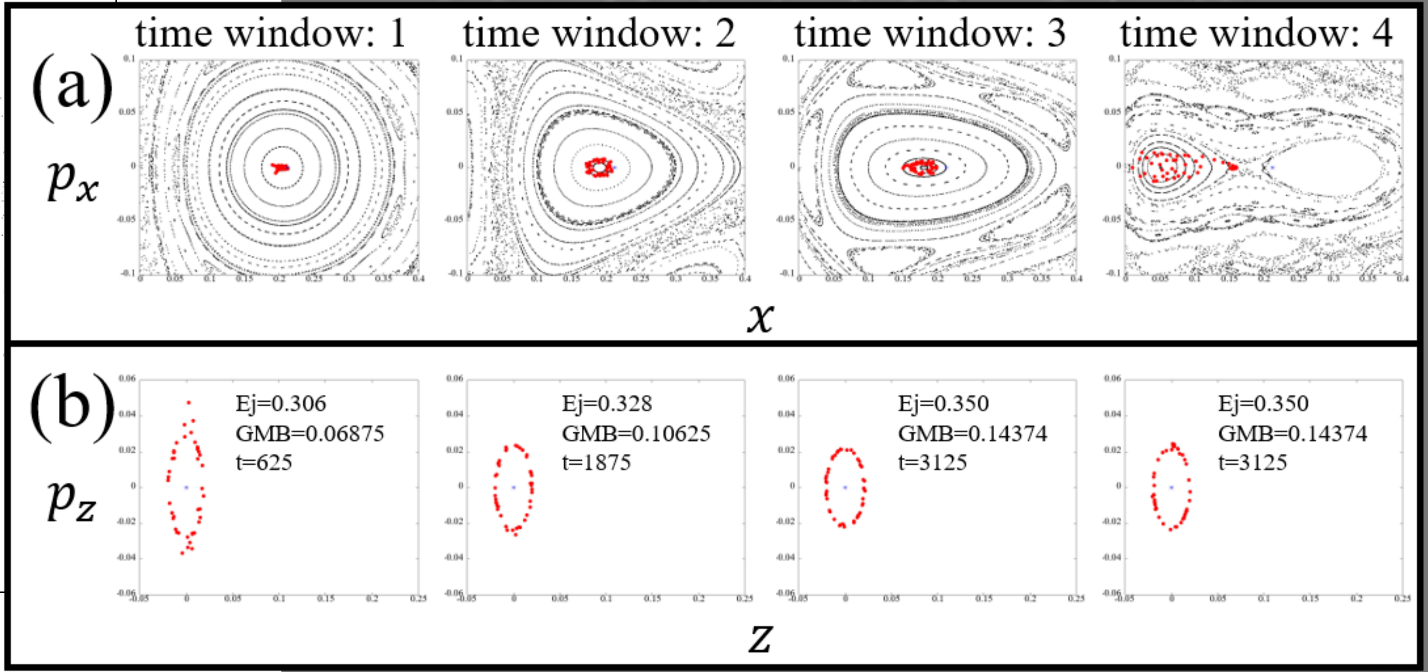


Figure 14. The (z, p_z) projection of the 4D phase space of x1 orbits perturbed in the p_z-direction, at $E_J = -0.41$. We indicate with black and grey '*' symbols the location of the x1v1 and x1v1′ p.o., respectively. Red and green '×' symbols indicate the two branches of x1v2. The '▲' at $(z, p_z) = (0.1, 0.42)$ marks the location of the p.o. x1mul2. Consequents corresponding to parts of the orbits that are plotted in other figures are marked with ⊙ symbols.



P&K14

Ευχαριστώ!!